AD A125328

CONTRACT REPORT ARBRL-CR-00502

DYNAMICS OF RIGID GUNS WITH STRAIGHT TUBES

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February 1983

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Contract Report ARBRL-CR-00502	2. GOVT ACCESSION NO. AD A125 328	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) DYNAMICS OF RIGID GUNS WITH STRAIGHT TUBES		5. TYPE OF REPORT & PERIOD COVERED INTERIM (Comp. Tasks 2) Nov 1980 - Nov 1981
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Henry L. Langhaar Arthur P. Boresi		8. CONTRACT OR GRANT NUMBER(s) DAAK11-80-C-0039
9. PERFORMING ORGANIZATION NAME AND ADDRESS BLM Applied Mechanics Associates 3310 Willett Drive Laramie, WY 82070		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1L161102AH43
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Armament Research & Development Command U.S. Army Ballistic Research Laboratory (DRDAR-BL) Aberdeen Proving Ground, MD 21005		12. REPORT DATE February 1983
		13. NUMBER OF PAGES 92
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Rigid Body Dynamics	Forces on Projectile	Transverse In-Bore Forces
Gun Dynamics	Projectile Dynamics	Transverse In-Bore Dynamics
Motion of Gun Tube	Forces on Gun Tube	
Projectile Motion	Transverse In-Bore Motion	

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Each of the four sections in this report deals with the motion, the forces, and the moments experienced by a rigid gun with a straight axis and an offset breech, and with a certain simple type of support that attaches the gun to a Galilean reference frame. The analysis treats the behavior of the system while the projectile is in the barrel. After the projectile leaves the muzzle, its motion and that of the gun are independent, and they may be analyzed by the dynamical theory of a single rigid body. The motion of the projectile with

(continued on next page)

20.   (continued)

respect to the tube is regarded as known. Also, the base pressures on the projectile and the tube are regarded as known functions of time. Geometrically, the projectile is regarded as a body of revolution that fits snuggly in the tube, so that there is no balloting. However, a lopsided distribution of mass density in the projectile is admitted in Sections 1, 2, and 3. In Section 4, only a balanced projectile is considered. In applications of the momentum principles, the momentum of gases in the tube is neglected. It can be introduced empirically by augmenting the mass of the projectile by a portion of the mass of the charge. This concept of effective mass has a precedent in other branches of fluid mechanics.

Section 1 treats a gun that is completely unsupported. The free gun is an extreme case; since the recoil mechanism and the trunnion would restrain the motion. Section 2 treats another extreme case; namely, a rigid gun that is completely fixed. In this case, there is no problem of motion, since the motion of the projectile is presumed to be given. Consequently, the analysis deals only with the forces and moments of interaction between the gun and the projectile. Section 3 treats the motion of a gun that translates freely along a guide, but it is constrained against rotation. Section 4 treats a gun with a trunnion and a recoil mechanism that reacts with a force that is an arbitrary function of the displacement and velocity of recoil. The motion is constrained to take place in a plane.

Vectorial mechanics is used in Sections 1, 2, and 3. In Section 4, the Lagrange equations (scalar form) are used.

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SECTION 1
FREE RIGID GUN TUBE WITH ECCENTRIC BREECH AND
DYNAMICALLY UNBALANCED PROJECTILE

1.1   INTRODUCTION

The motion of a free rigid gun in a gravitationless environment
is investigated in this section. Also, the force and the moment that the
projectile exerts on the gun are derived. Gravity, acting on a free gun,
would cause the center of mass of the system to descend with constant
acceleration g. The supports of an actual gun prevent a free fall of the
system. Consequently, it is questionable whether inclusion of gravity in
the analysis would make the mathematical model more realistic. Like gravity,
the resistance of air in the tube ahead of the projectile is an external
force. Its magnitude can be calculated by gas dynamics.[1] However, in this
chapter, the air resistance on the projectile is disregarded. The system,
as it is conceived, accordingly has no external forces acting on it. Since
the relative motion of the projectile with respect to the tube is assumed
to be given, the motion of the system is determined by the laws of conserva-
tion of rectilinear and angular momentum.

1.2   TERMINOLOGY AND NOTATIONS

The gun tube and the breech block together form a rigid body
called the "gun." The system consists of the gun and the projectile in the
tube. The point on the geometric axis of the projectile that lies closest
to the center of mass of the projectile is called the "geometric center" of
the projectile. The location of the center of mass of the projectile at the
initial instant is called the "starting point of the center of mass of the
projectile."

A bar over a letter denotes a vector. A caret over a letter
denotes a unit vector. A dot over a letter denotes the derivative with
respect to time t.   Ignition occurs at the instant t = 0. The following
notations are illustrated by Figure 1.

Point O is the center of mass of the whole system.

Point P is the center of mass of the gun.

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[1] R. Courant and K. Friedrichs, "Supersonic Flow and Shock Waves (U),"
Chap. III, Interscience Publishers, New York, 1948.
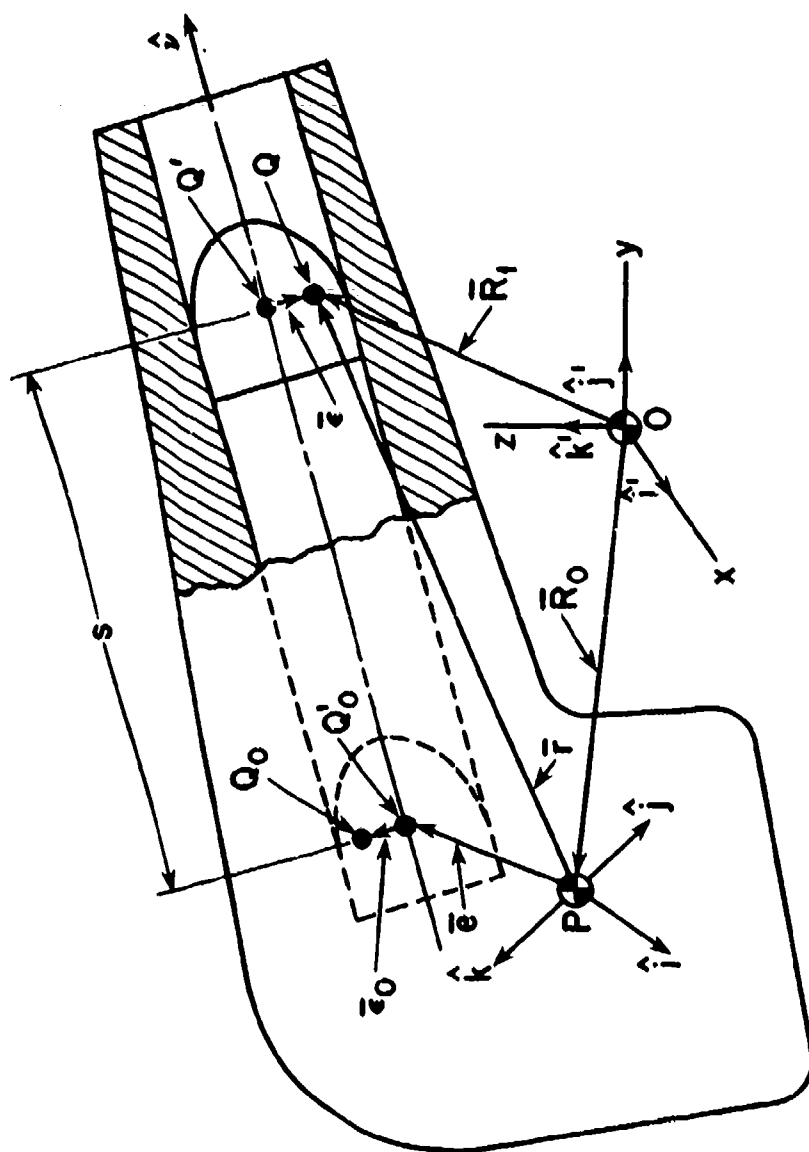
Figure 1. Gun System

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Point Q is the center of mass of the projectile at time t.

Point Q' is the geometric center of the projectile at time t.

Point Q_0 is the starting point of the center of mass of the projectile.

Point Q_0' is the initial location of point Q'.

F denotes a Galilean reference frame; e.g., the earth.

x,y,z are rectangular coordinates fixed in frame F.

$\hat{i}',\hat{j}',\hat{k}'$ are unit vectors along the axes x,y,z.

$\hat{i},\hat{j},\hat{k}$ are unit vectors along the principal axes of inertia of the gun through point P.

$\hat{a},\hat{b},\hat{c}$ are unit vectors along the principal axes of inertia of the projectile through point Q. None of them is necessarily parallel to the axis of the tube.

$\hat{a}_0,\hat{b}_0,\hat{c}_0$ are unit vectors along the principal axes of inertia of the projectile through point Q_0, when the projectile is in the starting position.

$\hat{v}$ is a unit vector along the axis of the tube.

$\overline{R}_0$ is the vector $\overrightarrow{OP}$.

$\overline{R}_1$ is the vector $\overrightarrow{OQ}$.

$\overline{r}$ is the vector $\overrightarrow{PQ}$; $\overline{r} = \overline{R}_1 - \overline{R}_0$.

$\overline{c}$ is the vector $\overrightarrow{PQ_0}$'.

$\overline{\epsilon}$ is the vector $\overrightarrow{Q'Q}$.

$\overline{\epsilon}_0$ is the vector $\overrightarrow{Q_0'Q_0}$.

s is the distance that the projectile has traveled relative to the tube at time t.

χ is the angle through which the projectile has turned relative to the tube at time t.

$\omega = \dot{\chi}$ is the spin (angular velocity) of the projectile relative to the tube at time t.

M is the mass of the gun.

m is the mass of the projectile.

I_1,I_2,I_3 are the principal moments of inertia of the gun with respect to its center of mass P.

i_1,i_2,i_3 are the principal moments of inertia of the projectile with respect to its center of mass Q.

$\overline{W}$ is the angular velocity of the gun relative to frame F.

$\overline{w}$ is the angular velocity of the projectile relative to frame F.

W_1, W_2, W_3 are components of $\overline{W}$, defined by $\overline{W} = \hat{i}W_1 + \hat{j}W_2 + \hat{k}W_3$.

w_1, w_2, w_3 are components of $\overline{w}$, defined by $\overline{w} = \hat{a}w_1 + \hat{b}w_2 + \hat{c}w_3$.

r_1, r_2, r_3 are components of $\overline{r}$, defined by $\overline{r} = \hat{i}r_1 + \hat{j}r_2 + \hat{k}r_3$.

X_0, Y_0, Z_0 are the (x,y,z) coordinates of the center of mass P of the gun; i.e., $\overline{R}_0 = \hat{i}'X_0 + \hat{j}'Y_0 + \hat{k}'Z_0$.

e_1, e_2, e_3 are components of the vector $\overline{e}$, defined by $\overline{e} = \hat{i}e_1 + \hat{j}e_2 + \hat{k}e_3$.

α, β, γ are direction cosines of the axis of the tube, defined by $\hat{v} = \hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma$.

$\epsilon_1, \epsilon_2, \epsilon_3$ are components of vector $\overline{\epsilon}_0$, defined by $\overline{\epsilon}_0 = \hat{i}\epsilon_1 + \hat{j}\epsilon_2 + \hat{k}\epsilon_3$.

ℓ_i, m_i, n_i are direction cosines of vectors $\hat{i}, \hat{j}, \hat{k}$ with respect to axes (x,y,z), defined by the matrix

	$\hat{i}'$	$\hat{j}'$	$\hat{k}'$
$\hat{i}$	ℓ_1	m_1	n_1
$\hat{j}$	ℓ_2	m_2	n_2
$\hat{k}$	ℓ_3	m_3	n_3

a_i, b_i, c_i are direction cosines of vectors $(\hat{a}, \hat{b}, \hat{c})$, defined by the matrix

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{a}$	a_1	a_2	a_3
$\hat{b}$	b_1	b_2	b_3
$\hat{c}$	c_1	c_2	c_3

$\alpha_i, \beta_i, \gamma_i$ are direction cosines of vectors $(\hat{a}_0, \hat{b}_0, \hat{c}_0)$, defined by the matrix

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{a}_0$	α_1	β_1	γ_1
$\hat{b}_0$	α_2	β_2	γ_2
$\hat{c}_0$	α_3	β_3	γ_3

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$\vec{H}$ is the angular momentum of the gun about point O.

$\vec{h}$ is the angular momentum of the projectile about point O.

$P_0(t)$ is the pressure of the gases on the breech.

$P_1(t)$ is the pressure of the gases on the base of the projectile.

A is the cross-sectional area of the bore.

$\vec{F}$ is the resultant force exerted by the projectile on the tube.

$\vec{M}$ is the moment about the center of mass of the gun of the contact forces that the projectile exerts on the tube.

$\vec{M}'$ is the moment about the center of mass of the projectile of the contact forces that the tube exerts on the projectile.

g is the scalar acceleration of gravity.

$\vec{g}$ is the vector acceleration of gravity (directed vertically downward).

R is the resisting force of the air ahead of the projectile

$$\mu = \frac{Mm}{M + m}$$

M_1, M_2, M_3 are components of vector $\vec{M}$, defined by $\vec{M} = \hat{i}M_1 + \hat{j}M_2 + \hat{k}M_3$.

F_1, F_2, F_3 are components of vector $\vec{F}$, defined by $\vec{F} = \hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3$.

$\vec{f} = \hat{i}f_1 + \hat{j}f_2 + \hat{k}f_3$ is the correction to force $\vec{F}$ to account for unbalance of the projectile.

1.3   CONSERVATION OF MOMENTUM

Since there are no external forces, the center of mass O of the system remains fixed in Frame F. Consequently, in this section, O is taken to be the origin of coordinates (x,y,z). At time t, the center of mass Q of the projectile lies at point $\vec{R}_1 = \vec{R}_0 + \vec{r}$ (Figure 1). Since the origin O is the center of mass of the system, and since momentum of the gases is neglected,

$$M\vec{R}_0 + m\vec{R}_1 = 0 \qquad\qquad (1.1)$$

Consequently,

$$M\ddot{\vec{R}}_0 + m(\ddot{\vec{R}}_0 + \ddot{\vec{r}}) = 0$$

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Therefore,

$$\bar{R}_0 = \frac{-m\bar{r}}{M + m} \; ; \; \bar{R}_1 = \frac{M\bar{r}}{M + m} = -\frac{M}{m}\bar{R}_0 \qquad (1.2)$$

Equation (1.2) signifies that the vectors $\bar{R}_0$ and $\bar{R}_1$ are collinear. Consequently, point O lies on the line PQ. Equation (1.2) yields

$$M\bar{R}_0 \times \dot{\bar{R}}_0 + m\bar{R}_1 \times \dot{\bar{R}}_1 = \frac{Mm}{M + m} \bar{r} \times \dot{\bar{r}} \qquad (1.3)$$

Equation (1.3) will be used later.

The direction cosines of vectors $(\hat{i}, \hat{j}, \hat{k})$ with respect to axes (x, y, z) are (ℓ_i, m_i, n_i) (see Notations). Also

$$\bar{R}_0 = \hat{i}'X_0 + \hat{j}'Y_0 + \hat{k}'Z_0$$

Equation (1.2) yields

$$\bar{R}_0 = \frac{-m}{M + m}(\hat{i}r_1 + \hat{j}r_2 + \hat{k}r_3)$$

Also,

$$\hat{i} = \ell_1\hat{i}' + m_1\hat{j}' + n_1\hat{k}'$$

$$\hat{j} = \ell_2\hat{i}' + m_2\hat{j}' + n_2\hat{k}'$$

$$\hat{k} = \ell_3\hat{i}' + m_3\hat{j}' + n_3\hat{k}'$$

Consequently,

$$X_0 = \frac{-m}{M + m}(r_1\ell_1 + r_2\ell_2 + r_3\ell_3)$$

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$$Y_0 = \frac{-m}{M + m}(r_1 m_1 + r_2 m_2 + r_3 m_3)$$

$$Z_0 = \frac{-m}{M + m}(r_1 n_1 + r_2 n_2 + r_3 n_3) \tag{1.4}$$

1.4 ANGULAR VELOCITY OF THE PROJECTILE

The absolute angular velocity of the projectile is

$$\bar{w} = \hat{a} w_1 + \hat{b} w_2 + \hat{c} w_3$$

Let $(\hat{a}', \hat{b}', \hat{c}')$ be an orthogonal triad of unit vectors, such that $\hat{c}' = \hat{v}$, where $\hat{v}$ is the unit vector along the axis of the tube. Then $\bar{w}$ may be written alternatively as follows:

$$\bar{w} = \hat{a}' w_1' + \hat{b}' w_2' + \hat{c}' w_3'$$

The angular velocity of the projectile relative to the tube is a vector of magnitude $\omega = \dot{\chi}$ that is coaxial with the tube. The variable ω is the spin; it is regarded as a known function of t. Accordingly, the $\hat{c}'$ component of the absolute angular velocity of the projectile is

$$w_3' = \omega + \bar{W} \cdot \hat{c}'$$

where $\bar{W}$ is the angular velocity of the gun. The $\hat{a}'$ and $\hat{b}'$ components of the absolute angular velocity of the projectile are the same as those of the tube, since balloting is excluded. Accordingly,

$$w_1' = \hat{a}' \cdot \bar{W} \; ; \; w_2' = \hat{b}' \cdot \bar{W}$$

Consequently[*],

$$\bar{w} = \hat{a}'\hat{a}' \cdot \bar{W} + \hat{b}'\hat{b}' \cdot \bar{W} + \hat{c}'\hat{c}' \cdot \bar{W} + \hat{c}'\omega$$

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[*] *The notation $\hat{a}'\hat{a}' \cdot \bar{W}$, etc., is short for $\hat{a}'(\hat{a}' \cdot \bar{W})$. Since $\hat{a}' \cdot \bar{W}$ is a scalar, $\hat{a}'\hat{a}' \cdot \bar{W}$ denotes multiplication of vector $\hat{a}'$ by the scalar $\hat{a}' \cdot \bar{W}$.*

However, the following relationship is an identity:

$$\hat{a}'\hat{a}' \cdot \overline{W} + \hat{b}'\hat{b}' \cdot \overline{W} + \hat{c}'\hat{c}' \cdot \overline{W} = \overline{W}$$

Also, $\hat{c}' = \hat{\nu}$. Consequently,

$$\overline{w} = \overline{W}' + \hat{\nu}\omega \tag{1.5}$$

Since,

$$\overline{w} = \hat{a}w_1 + \hat{b}w_2 + \hat{c}w_3 \; ; \; w_1 = \overline{w} \cdot \hat{a} \; ; \; w_2 = \overline{w} \cdot \hat{b} \; ; \; w_3 = \overline{w} \cdot \hat{c}$$

Also,

$$\hat{a} = \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3$$

$$\hat{b} = \hat{i}b_1 + \hat{j}b_2 + \hat{k}b_3$$

$$\hat{c} = \hat{i}c_1 + \hat{j}c_2 + \hat{k}c_3$$

Consequently, Eq. (1.5) yields

$$w_1 = a_1W_1 + a_2W_2 + a_3W_3 + (\alpha a_1 + \beta a_2 + \gamma a_3)\omega$$

$$w_2 = b_1W_1 + b_2W_2 + b_3W_3 + (\alpha b_1 + \beta b_2 + \gamma b_3)\omega$$

$$w_3 = c_1W_1 + c_2W_2 + c_3W_3 + (\alpha c_1 + \beta c_2 + \gamma c_3)\omega \tag{1.6}$$

1.5   THE VECTORS $\hat{a}, \hat{b}, \hat{c}, \overline{\epsilon}$

So far, the vectors $\hat{a}$, $\hat{b}$, $\hat{c}$, whose componer   ppear in Eq. (1.6), are undetermined. The displacement vector field of a igid body that under-goes a rotation about an axis that is oblique to the coordinate ax ; is derived in Appendix A. Vectors $\hat{a}$, $\hat{b}$, $\hat{c}$ are imbedded in the projectile and

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move with it. Accordingly, Eq. (A-2) in Appendix A applies to them. The vector $\bar{\rho}$ in Eq. (A-2) may be given various special designations. If $\bar{\rho} = \hat{a}_0$, $\hat{a} = \bar{R}$. If $\bar{\rho} = \hat{b}_0$, $\hat{b} = \bar{R}$, and likewise for $\hat{c}_0$ and $\hat{c}$. Consequently[*],

$$\hat{a} = \hat{\nu} \times \hat{a}_0 \sin\chi + \hat{\nu}\hat{\nu} \cdot \hat{a}_0 (1 - \cos\chi) + \hat{a}_0 \cos\chi$$

$$\hat{b} = \hat{\nu} \times \hat{b}_0 \sin\chi + \hat{\nu}\hat{\nu} \cdot \hat{b}_0 (1 - \cos\chi) + \hat{b}_0 \cos\chi$$

$$\hat{c} = \hat{\nu} \times \hat{c}_0 \sin\chi + \hat{\nu}\hat{\nu} \cdot \hat{c}_0 (1 - \cos\chi) + \hat{c}_0 \cos\chi \tag{1.7}$$

It can be shown directly from Eq. (1.7) that $(\hat{a}, \hat{b}, \hat{c})$ are an orthogonal triad of unit vectors, as they should be. Equation (1.7) also yields

$$\hat{\nu} \cdot \hat{a} = \hat{\nu} \cdot \hat{a}_0 \; ; \; \hat{\nu} \cdot \hat{b} = \hat{\nu} \cdot \hat{b}_0 \; ; \; \hat{\nu} \cdot \hat{c} = \hat{\nu} \cdot \hat{c}_0 \tag{1.8}$$

Equation (1.8) reflects the fact that a rotation of a body about an axis does not change the angle between the axis and any line that is scribed in the body.

The direction cosines of the vectors $(\hat{a}_0, \hat{b}_0, \hat{c}_0)$ are given by the following matrix:

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{a}_0$	α_1	β_1	γ_1
$\hat{b}_0$	α_2	β_2	γ_2
$\hat{c}_0$	α_3	β_3	γ_3
$\hat{\nu}$	α	β	γ

(1.9)

The direction cosines in Eq. (1.9) are known constants. Equations (1.7) and (1.9) yield

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$$\hat{a} = \hat{i}\,[(\beta\gamma_1 - \gamma\beta_1)\sin\chi + \alpha(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1)(1 - \cos\chi) + \alpha_1\cos\chi]$$

$$+ \hat{j}\,[(\gamma\alpha_1 - \alpha\gamma_1)\sin\chi + \beta(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1)(1 - \cos\chi) + \beta_1\cos\chi]$$

$$+ \hat{k}\,[(\alpha\beta_1 - \beta\alpha_1)\sin\chi + \gamma(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1)(1 - \cos\chi) + \gamma_1\cos\chi]$$

$$\hat{b} = \hat{i}\,[(\beta\gamma_2 - \gamma\beta_2)\sin\chi + \alpha(\alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2)(1 - \cos\chi) + \alpha_2\cos\chi]$$

$$+ \hat{j}\,[(\gamma\alpha_2 - \alpha\gamma_2)\sin\chi + \beta(\alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2)(1 - \cos\chi) + \beta_2\cos\chi]$$

$$+ \hat{k}\,[(\alpha\beta_2 - \beta\alpha_2)\sin\chi + \gamma(\alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2)(1 - \cos\chi) + \gamma_2\cos\chi]$$

$$\hat{c} = \hat{i}\,[(\beta\gamma_3 - \gamma\beta_3)\sin\chi + \alpha(\alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3)(1 - \cos\chi) + \alpha_3\cos\chi]$$

$$+ \hat{j}\,[(\gamma\alpha_3 - \alpha\gamma_3)\sin\chi + \beta(\alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3)(1 - \cos\chi) + \beta_3\cos\chi]$$

$$+ \hat{k}\,[(\alpha\beta_3 - \beta\alpha_3)\sin\chi + \gamma(\alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3)(1 - \cos\chi) + \gamma_3\cos\chi] \qquad (1.10)$$

Equation (1.10) determines the direction cosines (a_i, b_i, c_i) of vectors $(\hat{a}, \hat{b}, \hat{c})$ as functions of t.

If $\bar{\rho}$ is set equal to $\bar{\epsilon}_0$ in Eq. (A-2), $\bar{R} = \bar{\epsilon}$. Since $\bar{\epsilon}_0$ is perpendicular to $\hat{\nu}$, $\hat{\nu} \cdot \bar{\epsilon}_0 = 0$. Consequently, by Eq. (A-2),

$$\bar{\epsilon} = \hat{\nu} \times \bar{\epsilon}_0 \sin\chi + \bar{\epsilon}_0 \cos\chi \qquad (1.11)$$

Since $\bar{\epsilon}_0 \cdot \hat{\nu} = 0$, it follows from Eq. (1.11) that $\bar{\epsilon} \cdot \hat{\nu} = 0$, as it should be. Also, Eq. (1.11) yields

$$\bar{\epsilon} \cdot \bar{\epsilon} = \bar{\epsilon}_0 \cdot \bar{\epsilon}_0 = \epsilon^2 \qquad (1.12)$$

where ϵ is the eccentricity of the projectile.

1.6 THE VECTOR PRODUCT $\bar{r} \times \dot{\bar{r}}$

The vector product $\bar{r} \times \dot{\bar{r}}$ occurs in the expression for the angular momentum of the system about point O. By Figure 1,

$$\bar{r} = \bar{e} + \hat{\nu}s + \bar{\epsilon} \tag{1.13}$$

Hence,

$$\dot{\bar{r}} = \dot{\bar{e}} + \dot{\hat{\nu}}s + \hat{\nu}\dot{s} + \dot{\bar{\epsilon}}$$

In the reference frame of the gun, $\bar{e}$ and $\hat{\nu}$ are constant vectors. Consequently,

$$\dot{\bar{e}} = \bar{W} \times \bar{e} \; ; \; \dot{\hat{\nu}} = \bar{W} \times \hat{\nu} \tag{1.14}$$

Therefore,

$$\dot{\bar{r}} = \bar{W} \times \bar{e} + \bar{W} \times \hat{\nu}s + \hat{\nu}\dot{s} + \dot{\bar{\epsilon}} \tag{1.15}$$

Since $\dot{\bar{\epsilon}}_0 = \bar{W} \times \bar{\epsilon}_0$, Eq. (1.11) yields (see footnote, page 13)

$$\dot{\bar{\epsilon}} = \hat{\nu}\bar{W} \cdot \bar{\epsilon}_0 \sin\chi - \bar{\epsilon}_0\hat{\nu} \cdot \bar{W}\sin\chi + \omega\hat{\nu} \times \bar{\epsilon}_0\cos\chi + \bar{W} \times \bar{\epsilon}_0\cos\chi - \omega\bar{\epsilon}_0\sin\chi \tag{1.16}$$

Equations (1.11), (1.13), (1.15), and (1.16) yield

$$\bar{r} = \bar{e} + \hat{\nu}s + \hat{\nu} \times \bar{\epsilon}_0\sin\chi + \bar{\epsilon}_0\cos\chi \tag{1.17}$$

$$\dot{\bar{r}} = \bar{W} \times \bar{e} + \bar{W} \times \hat{\nu}s + \hat{\nu}\dot{s} + \hat{\nu}\bar{W} \cdot \bar{\epsilon}_0\sin\chi - \bar{\epsilon}_0\hat{\nu} \cdot \bar{W}\sin\chi$$

$$+ \omega\hat{\nu} \times \bar{\epsilon}_0\cos\chi + \bar{W} \times \bar{\epsilon}_0\cos\chi - \omega\bar{\epsilon}_0\sin\chi \tag{1.18}$$

Equation (1.18) may be written as follows:

$$\dot{\bar{r}} = \bar{W} \times \bar{r} + \hat{\nu}\dot{s} + \omega\hat{\nu} \times \bar{\epsilon}_0\cos\chi - \omega\bar{\epsilon}_0\sin\chi \tag{1.19}$$

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Hence (see footnote, page 13),

$$\bar{r} \times \dot{\bar{r}} = \bar{W}r^2 - \bar{r}\,\bar{r} \cdot \bar{W} + \bar{r} \times \hat{v}\dot{s} + \omega v \hat{\bar{r}} \cdot \bar{\epsilon}_0 \cos\chi - \omega \bar{\epsilon}_0 \bar{r} \cdot \hat{v}\cos\chi$$

$$- \omega\bar{r} \times \bar{\epsilon}_0 \sin\chi \tag{1.20}$$

With

$$\bar{r} = \hat{i}r_1 + \hat{j}r_2 + \hat{k}r_3 \;,\; \bar{W} = \hat{i}W_1 + \hat{j}W_2 + \hat{k}W_3 \;,$$

$$\hat{v} = \hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma \;,\; \text{and} \; \bar{\epsilon}_0 = \hat{i}\epsilon_1 + \hat{j}\epsilon_2 + \hat{k}\epsilon_3$$

Eq. (1.20) yields

$$\bar{r} \times \dot{\bar{r}} = \hat{i}\{(r_2^2 + r_3^2)W_1 - r_1 r_2 W_2 - r_1 r_3 W_3 + (\gamma r_2 - \beta r_3)\dot{s}$$

$$+ [\alpha(r_2\epsilon_2 + r_3\epsilon_3) - \epsilon_1(\beta r_2 + \gamma r_3)]\omega\cos\chi$$

$$- (r_2\epsilon_3 - r_3\epsilon_2)\omega\sin\chi\} + \hat{j}\{-r_2 r_1 W_1 + (r_3^2 + r_1^2)W_2$$

$$- r_2 r_3 W_3 + (\alpha r_3 - \gamma r_1)\dot{s} + [\beta(r_3\epsilon_3 + r_1\epsilon_1)$$

$$- \epsilon_2(\gamma r_3 + \alpha r_1)]\omega\cos\chi - (r_3\epsilon_1 - r_1\epsilon_3)\omega\sin\chi\}$$

$$+ \hat{k}\{-r_3 r_1 W_1 - r_3 r_2 W_2 + (r_1^2 + r_2^2)W_3 + (\beta r_1 - \alpha r_2)\dot{s}$$

$$+ [\gamma(r_1\epsilon_1 + r_2\epsilon_2) - \epsilon_3(\alpha r_1 + \beta r_2)]\omega\cos\chi$$

$$- (r_1\epsilon_2 - r_2\epsilon_1)\omega\sin\chi\} \tag{1.21}$$

1.7   ANGULAR MOMENTUM

The angular momentum of the gun with respect to a nonrotating observer at point P is[2]

$$\hat{i} I_1 W_1 + \hat{j} I_2 W_2 + \hat{k} I_3 W_3$$

Consequently, by Eq. (B-3) in Appendix B, the angular momentum of the gun with respect to point O is

$$\overline{H} = \hat{i} I_1 W_1 + \hat{j} I_2 W_2 + \hat{k} I_3 W_3 + M \overline{R}_0 \times \dot{\overline{R}}_0 \qquad (1.22)$$

Similarly, the angular momentum of the projectile about point O is

$$\overline{h} = \hat{a} i_1 w_1 + \hat{b} i_2 w_2 + \hat{c} i_3 w_3 + m \overline{R}_1 \times \dot{\overline{R}}_1 \qquad (1.23)$$

The vectors $\hat{a}$, $\hat{b}$, $\hat{c}$ are resolved into $\hat{i}$, $\hat{j}$, $\hat{k}$ components by the equations,

$$\hat{a} = \hat{i} a_1 + \hat{j} a_2 + \hat{k} a_3$$

$$\hat{b} = \hat{i} b_1 + \hat{j} b_2 + \hat{k} b_3$$

$$\hat{c} = \hat{i} c_1 + \hat{j} c_2 + \hat{k} c_3$$

Consequently,

$$\overline{h} = \hat{i}(a_1 i_1 w_1 + b_1 i_2 w_2 + c_1 i_3 w_3) + \hat{j}(a_2 i_1 w_1 + b_2 i_2 w_2 + c_2 i_3 w_3)$$

$$+ \hat{k}(a_3 i_1 w_1 + b_3 i_2 w_2 + c_3 i_3 w_3) + m \overline{R}_1 \times \dot{\overline{R}}_1 \qquad (1.24)$$

Consequently, in view of Eqs. (1.3) and (1.22),

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[2] *H. L. Langhaar and A. P. Boresi, Engineering Mechanics-Dynamics (U), McGraw-Hill, New York, 1959, Art. 15-10.*

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$$\overline{H} + \overline{h} = \hat{i}(I_1W_1 + a_1i_1w_1 + b_1i_2w_2 + c_1i_3w_3)$$

$$+ \hat{j}(I_2W_2 + a_2i_1w_1 + b_2i_2w_2 + c_2i_3w_3)$$

$$+ \hat{k}(I_3W_3 + a_3i_1w_1 + b_3i_2w_2 + c_3i_3w_3) + \frac{mM}{M+m}\,\overline{r} \times \dot{\overline{r}} \quad (1.25)$$

The term $\overline{r} \times \dot{\overline{r}}$ is expressed in the form $(-)\hat{i} + (-)\hat{j} + (-)\hat{k}$ by Eq. (1.21). Consequently, Eq. (1.25) represents $\overline{H} + \overline{h}$ in the form

$$\overline{H} + \overline{h} = \hat{i}A + \hat{j}B + \hat{k}C \qquad\qquad (1.26)$$

By the law of conservation of angular momentum, $\overline{H} + \overline{h}$ is a constant vector. However, in general, (A,B,C) are not constants, since the gun moves and accordingly $(\hat{i},\hat{j},\hat{k})$ are time-dependent vectors. The constant vectors $(\hat{i}',\hat{j}', \hat{k}')$ may be introduced by means of the equations,

$$\hat{i} = \ell_1\hat{i}' + m_1\hat{j}' + n_1\hat{k}'$$

$$\hat{j} = \ell_2\hat{i}' + m_2\hat{j}' + n_2\hat{k}'$$

$$\hat{k} = \ell_3\hat{i}' + m_3\hat{j}' + n_3\hat{k}'$$

Thus, $\overline{H} + \overline{h}$ is expressed as follows:

$$\overline{H} + \overline{h} = \hat{i}'(\ell_1A + \ell_2B + \ell_3C) + \hat{j}'(m_1A + m_2B + m_3C) + \hat{k}'(n_1A$$

$$+ n_2B + n_3C)$$

Accordingly,

$$\ell_1A + \ell_2B + \ell_3C = K_1$$

$$m_1A + m_2B + m_3C = K_2$$

$$n_1A + n_2B + n_3C = K_3 \tag{1.27}$$

where K_1, K_2, K_3 are constants. However, Eq. (1.27) introduces a complication, since (ℓ_i, m_i, n_i) are unknown functions of time.

A great simplification occurs if the system is initially at rest, since then $\overline{H} + \overline{h} = 0$ and $K_1 = K_2 = K_3 = 0$. Consequently, $A = B = C = 0$. Attention is restricted to this case.

Provisionally, the following notation is introduced:

$$\frac{Mm}{M + m}\,\overline{r} \times \dot{\overline{r}} = \hat{i}X + \hat{j}Y + \hat{k}Z \tag{1.28}$$

The terms X, Y, Z are given by Eq. (1.21). With Eqs. (1.25) and (1.28), the law of conservation of angular momentum, $\overline{H} + \overline{h} = 0$, yields

$$I_1 W_1 + a_1 i_1 w_1 + b_1 i_2 w_2 + c_1 i_3 w_3 + X = 0$$

$$I_2 W_2 + a_2 i_1 w_1 + b_2 i_2 w_2 + c_2 i_3 w_3 + Y = 0$$

$$I_3 W_3 + a_3 i_1 w_1 + b_3 i_2 w_2 + c_3 i_3 w_3 + Z = 0 \tag{1.29}$$

Now w_1, w_2, w_3 are given by Eq. (1.6), and X, Y, Z are given by Eq. (1.21). For brevity, the following notation is introduced:

$$\mu = \frac{Mm}{M + m} \tag{1.30}$$

Thus, the following equations are obtained from Eq. (1.29):

$$[I_1 + a_1^2 i_1 + b_1^2 i_2 + c_1^2 i_3 + \mu(r_2^2 + r_3^2)]W_1 + [a_1 a_2 i_1$$

$$+ b_1 b_2 i_2 + c_1 c_2 i_3 - \mu r_1 r_2]W_2 + [a_1 a_3 i_1 + b_1 b_3 i_2 + c_1 c_3 i_3$$

$$- \mu r_1 r_3]W_3 = -a_1 i_1 (\alpha a_1 + \beta a_2 + \gamma a_3)\omega - b_1 i_2 (\alpha b_1 + \beta b_2 + \gamma b_3)\omega$$

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$$- c_1 i_3 (\alpha c_1 + \beta c_2 + \gamma c_3)\omega - \mu(\gamma r_2 - \beta r_3)\dot{s}$$

$$- \mu\alpha(r_2\epsilon_2 + r_3\epsilon_3)\omega\cos\chi + \mu\epsilon_1(\beta r_2 + \gamma r_3)\omega\cos\chi$$

$$+ \mu(r_2\epsilon_3 - r_3\epsilon_2)\omega\sin\chi$$

$$[a_1 a_2 i_1 + b_1 b_2 i_2 + c_1 c_2 i_3 - \mu r_1 r_2]W_1 + [I_2 + a_2^2 i_1 + b_2^2 i_2$$

$$+ c_2^2 i_3 + \mu(r_3^2 + r_1^2)]W_2 + [a_2 a_3 i_1 + b_2 b_3 i_2 + c_2 c_3 i_3 - \mu r_2 r_3]W_3$$

$$= -a_2 i_1(\alpha a_1 + \beta a_2 + \gamma a_3)\omega - b_2 i_2(\alpha b_1 + \beta b_2 + \gamma b_3)\omega$$

$$- c_2 i_3(\alpha c_1 + \beta c_2 + \gamma c_3)\omega - \mu(\alpha r_3 - \gamma r_1)\dot{s} - \mu\beta(r_3\epsilon_3$$

$$+ r_1\epsilon_1)\omega\cos\chi + \mu\epsilon_2(\gamma r_3 + \alpha r_1)\omega\cos\chi + \mu(r_3\epsilon_1 - r_1\epsilon_3)\omega\sin\chi$$

$$[a_3 a_1 i_1 + b_3 b_1 i_2 + c_3 c_1 i_3 - \mu r_3 r_1]W_1 + [a_2 a_3 i_1 + b_2 b_3 i_2$$

$$+ c_2 c_3 i_3 - \mu r_2 r_3]W_2 + [I_3 + a_3^2 i_1 + b_3^2 i_2 + c_3^2 i_3$$

$$+ \mu(r_1^2 + r_2^2)]W_3 = -a_3 i_1(\alpha a_1 + \beta a_2 + \gamma a_3)\omega - b_3 i_2(\alpha b_1 + \beta b_2$$

$$+ \gamma b_3)\omega - c_3 i_3(\alpha c_1 + \beta c_2 + \gamma c_3)\omega$$

$$- \mu(\beta r_1 - \alpha r_2)\dot{s} - \mu\gamma(r_1\epsilon_1 + r_2\epsilon_2)\omega\cos\chi$$

$$+ \mu\epsilon_3(\alpha r_1 + \beta r_2)\omega\cos\chi + \mu(r_1\epsilon_2 - r_2\epsilon_1)\omega\sin\chi$$

$$(1.31)$$

The unknowns in Eq. (1.31) are W_1, W_2, W_3. The quantities I_1, I_2, I_3, i_1, i_2, i_3, μ, α, β, γ, ϵ_1, ϵ_2, ϵ_3 are known constants. The

quantities s, χ, ω are regarded as known functions of t. Perhaps the simplest way to determine χ is by the relationship,

$$\chi = \int_0^t \omega \, dt \qquad (1.32)$$

Since e_1, e_2, e_3 are known constants, Eq. (1.17) determines r_1, r_2, r_3 as functions of t. By Eq. (1.10), the $(\hat{i}, \hat{j}, \hat{k})$ components (a_i, b_i, c_i) of vectors $(\hat{a}, \hat{b}, \hat{c})$ are known functions of t. With a computer program, Eq. (1.31) can be solved for any sequence of values of t that covers the period in which the projectile is in the tube. Thus, the functions $W_1(t)$, $W_2(t)$, $W_3(t)$ can be tabulated and plotted.

1.8 MOTION OF THE GUN

A knowledge of the functions $W_1(t)$, $W_2(t)$, $W_3(t)$ does not immediately determine the motion of the gun in reference frame F. It is necessary to determine the direction cosines (ℓ_i, m_i, n_i) as functions of t. When these functions are known, the motion of the center of mass of the gun is determined by Eqs. (1.4) and (1.17), and the absolute orientations of the principal axes of the gun are determined as functions of t.

The problem of determining the functions ℓ_i, m_i, n_i is purely one of kinematics of a rigid body. The results are[3]

$$\dot{\ell}_1 = W_3 \ell_2 - W_2 \ell_3 \;\; ; \;\; \dot{\ell}_2 = W_1 \ell_3 - W_3 \ell_1 \;\; ; \;\; \dot{\ell}_3 = W_2 \ell_1 - W_1 \ell_2 \;\; ;$$

$$\dot{m}_1 = W_3 m_2 - W_2 m_3 \;\; ; \;\; \dot{m}_2 = W_1 m_3 - W_3 m_1 \;\; ; \;\; \dot{m}_3 = W_2 m_1 - W_1 m_2 \;\; ;$$

$$\dot{n}_1 = W_3 n_2 - W_2 n_3 \;\; ; \;\; \dot{n}_2 = W_1 n_3 - W_3 n_1 \;\; ; \;\; \dot{n}_3 = W_2 n_1 - W_1 n_2 \qquad (1.33)$$

Since the functions W_1, W_2, W_3 have been determined, Eqs. (1.33) are differential equations that determine the functions ℓ_i, m_i, n_i when the initial values are given.

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[3] H. L. Langhaar, *Energy Methods in Applied Mechanics (U)*, John Wiley and Sons, New York, 1962, Chap. 7.

Instead of working with the nine unknown functions ℓ_i, m_i, n_i, it is possible to work with three Euler angles (θ, ϕ, ψ) by means of the following relations[3]:

$$\ell_1 = \sin\phi\sin\psi - \cos\theta\cos\phi\cos\psi$$

$$m_1 = -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi$$

$$n_1 = \sin\theta\cos\psi$$

$$\ell_2 = \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi$$

$$m_2 = -\cos\phi\cos\psi + \cos\theta\sin\phi\sin\psi$$

$$n_2 = -\sin\theta\sin\psi$$

$$\ell_3 = \sin\theta\cos\phi$$

$$m_3 = \sin\theta\sin\phi$$

$$n_3 = \cos\theta \tag{1.34}$$

The differential equations for θ, ϕ, ψ are[3]

$$W_1 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$W_2 = -\dot{\theta}\cos\psi - \dot{\phi}\sin\theta\sin\psi$$

$$W_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{1.35}$$

Equations (1.35) are easily solved algebraically for the derivatives. Thus,

$$\frac{d\theta}{dt} = -W_1 \sin\psi - W_2 \cos\psi$$

$$\frac{d\phi}{dt} = (W_1 \cos\psi - W_2 \sin\psi) \csc\theta$$

$$\frac{d\psi}{dt} = W_3 - (W_1 \cos\psi - W_2 \sin\psi) \cot\theta \tag{1.36}$$

According to an existence theorem in the theory of ordinary first-order differential equations,[4] there is a unique solution $\theta(t)$, $\phi(t)$, $\psi(t)$ of Eq. (1.36) which takes given initial values $(\theta_0, \phi_0, \psi_0)$, provided that θ avoids the singular values, $\theta = 0$ and $\theta = \pi$. After $\theta(t)$, $\phi(t)$, $\psi(t)$ are determined, (ℓ_i, m_i, n_i) are determined by Eq. (1.34).

The existence theorem also applies directly to Eq. (1.33), i.e., there are unique functions $\ell_i(t)$, $m_i(t)$, $n_i(t)$ that satisfy Eq. (1.33) and that take given initial values (ℓ_i^0, m_i^0, n_i^0). Although Eq. (1.33) involves nine dependent variables, it has the advantage that the equations are linear.

A numerical solution of the differential equations appears feasible. It is not necessary to project far into the future, since the projectile quickly leaves the muzzle.

1.9   CASE OF A BALANCED PROJECTILE

If the projectile is perfectly balanced $i_1 = i_2$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$. Also $c_1 = \alpha$, $c_2 = \beta$, $c_3 = \gamma$. Then the first of Eqs. (1.31) becomes

$$[I_1 + i_1(a_1^2 + b_1^2) + \alpha^2 i_3 + \mu(r_2^2 + r_3^2)]W_1$$

$$+ [i_1(a_1 a_2 + b_1 b_2) + \alpha\beta i_3 - \mu r_1 r_2]W_2$$

$$+ [i_1(a_1 a_3 + b_1 b_3) + \alpha\gamma i_3 - \mu r_1 r_3]W_3$$

[4] E. L. Ince, *Ordinary Differential Equations (U)*, Dover Pubs., New York, 1944, Arts. 3.3 and 3.31.

$$= -i_1\omega[\alpha a_1^2 + \beta a_1 a_2 + \gamma a_1 a_3 + \alpha b_1^2 + \beta b_1 b_2 + \gamma b_1 b_3]$$

$$- \alpha i_3(\alpha^2 + \beta^2 + \gamma^2)\omega - \mu(\gamma r_2 - \beta r_3)\dot{s}$$

In view of identities among the direction cosines, this reduces to

$$[I_1 + i_1 + \alpha^2(i_3 - i_1) + \mu(r_2^2 + r_3^2)]W_1 + [\alpha\beta(i_3 - i_1)$$

$$- \mu r_1 r_2]W_2 + [\alpha\gamma(i_3 - i_1) - \mu r_1 r_3]W_3 = -\alpha i_3\omega - \mu(\gamma r_2 - \beta r_3)\dot{s}$$

$$(1.37)$$

Likewise from the second and third of Eqs. (1.31),

$$[\alpha\beta(i_3 - i_1) - \mu r_1 r_2]W_1 + [I_2 + i_1 + \beta^2(i_3 - i_1)$$

$$+ \mu(r_3^2 + r_1^2)]W_2 + [\beta\gamma(i_3 - i_1) - \mu r_2 r_3]W_3$$

$$= -\beta i_3\omega - \mu(\alpha r_3 - \gamma r_1)\dot{s} \qquad (1.37)$$

$$[\alpha\gamma(i_3 - i_1) - \mu r_3 r_1]W_1 + [\beta\gamma(i_3 - i_1) - \mu r_3 r_2]W_2$$

$$+ [I_3 + i_1 + \gamma^2(i_3 - i_1) + \mu(r_1^2 + r_2^2)]W_3$$

$$= -\gamma i_3\omega - \mu(\beta r_1 - \alpha r_2)\dot{s} \qquad (1.37)$$

Equation (1.37) is a simplified form of Eq. (1.31) that applies only if the projectile is perfectly balanced.

1.10   ELEMENTARY RELATIONS BETWEEN FORCES AND MOMENTS

The preceding theory deals with the motion of a free rigid gun. The forces and moments of interaction between the projectile and the gun also are of interest. Although gravity has been neglected, it would have only a small effect on these forces and moments.

The resultant force of contact that the projectile exerts on the tube is designated as $\overline{F}$. The force of contact that the tube exerts on the projectile is $-\overline{F}$. If the gun is regarded as a free body, the net external force on it is

$$\overline{F} - Ap_0\hat{\nu} \tag{1.38}$$

If the projectile is regarded as a free body, the net external force on it (neglecting resistance of air ahead of the projectile) is

$$-\overline{F} + Ap_1\hat{\nu} \tag{1.39}$$

The detailed forces of contact that the projectile exerts on the tube are designated as $\overline{f}_1$, $\overline{f}_2$, $\overline{f}_3$, . . . Hence,

$$\overline{F} = \Sigma\overline{f}_i \tag{1.40}$$

Let force $\overline{f}_i$ act at the point $\overline{r} + \overline{\lambda}_i$, where $\overline{r}$ is the vector from the center of mass of the gun to the center of mass of the projectile. Then the moment that the forces $\overline{f}_i$ exert about the center of mass of the gun is

$$\overline{M} = \Sigma(\overline{r} + \overline{\lambda}_i) \times \overline{f}_i = \overline{r} \times \overline{F} + \Sigma\overline{\lambda}_i \times \overline{f}_i \tag{a}$$

The forces of contact that the tube exerts on the projectile are $-\overline{f}_1$, $-\overline{f}_2$, $-\overline{f}_3$, . . . The moment of these forces about the center of mass of the projectile is

$$\overline{M}' = - \Sigma\overline{\lambda}_i \times \overline{f}_i \tag{b}$$

By Eqs. (a) and (b),

$$\overline{M}' = \overline{r} \times \overline{F} - \overline{M} \tag{1.41}$$

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The moment about the center of mass of the gun of all external forces that act on the gun is

$$\overline{M} - \overline{e} \times \hat{v}p_0A \tag{1.42}$$

where $\overline{e}$ is the vector from the center of mass of the gun to the initial location of the geometric center of the projectile. The moment about the center of mass of the projectile of all the forces that act on the projectile is

$$\overline{M}' - \overline{\epsilon} \times \hat{v}p_1A \tag{1.43}$$

where $\overline{\epsilon}$ is the vector from the geometric center of the projectile to the center of mass of the projectile. If the center of mass of the projectile is on the axis of the tube, $\overline{\epsilon} = 0$.

1.11   MOMENT OF FORCES ON THE GUN

The resultant moment $\overline{M}$ exerted on the gun by the forces of contact with the projectile is resolved into components along the principal axes of inertia of the gun through its center of mass, i.e.,

$$\overline{M} = \hat{i}M_1 + \hat{j}M_2 + \hat{k}M_3$$

Also, by definition,

$$\hat{v} = \hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma$$

$$\overline{e} = \hat{i}e_1 + \hat{j}e_2 + \hat{k}e_3 \tag{1.44}$$

Consequently, by Eq. (1.42), the net moment about the center of mass of the gun is

$$\hat{i}[M_1 - p_0A(\gamma e_2 - \beta e_3)] + \hat{j}[M_2 - p_0A(\alpha e_3 - \gamma e_1)]$$

$$+ \hat{k}[M_3 - p_0A(\beta e_1 - \alpha e_2)] \tag{1.45}$$

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The components M_i are determined by Euler's dynamical equations for a rigid body[3]; namely,

$$I_1 \frac{dW_1}{dt} - (I_2 - I_3)W_2 W_3 + p_0 A(\gamma e_2 - \beta e_3) = M_1$$

$$I_2 \frac{dW_2}{dt} - (I_3 - I_1)W_3 W_1 + p_0 A(\alpha e_3 - \gamma e_1) = M_2$$

$$I_3 \frac{dW_3}{dt} - (I_1 - I_2)W_1 W_2 + p_0 A(\beta e_1 - \alpha e_2) = M_3 \qquad (1.46)$$

Since (W_1, W_2, W_3) are obtained from a computer program for the solution of Eq. (1.31) (or, in the case of a balanced projectile, from Eq. (1.37)), the derivatives dW_i/dt can be obtained by numerical differentiation. The quantities (α, β, γ), (e_1, e_2, e_3), and (I_1, I_2, I_3) are known constants. Consequently, Eq. (1.46) determines $\overline{M}$, provided that the breech pressure $p_0(t)$ is known.

1.12 FORCE OF A BALANCED PROJECTILE ON THE TUBE

The angular acceleration of the gun is $\dot{\overline{W}}$. Since $\overline{W} = \hat{i}W_1 + \hat{j}W_2 + \hat{k}W_3$,

$$\dot{\overline{W}} = \hat{i}\dot{W}_1 + \hat{j}\dot{W}_2 + \hat{k}\dot{W}_3 + \dot{\hat{i}}W_1 + \dot{\hat{j}}W_2 + \dot{\hat{k}}W_3$$

Also,

$$\dot{\hat{i}} = \overline{W} \times \hat{i} = \hat{j}W_3 - \hat{k}W_2$$

$$\dot{\hat{j}} = \overline{W} \times \hat{j} = \hat{k}W_1 - \hat{i}W_3$$

$$\dot{\hat{k}} = \overline{W} \times \hat{k} = \hat{i}W_2 - \hat{j}W_1$$

Consequently,

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$$\hat{i}\dot{W}_1 \cdot \hat{j}\dot{W}_2 + \hat{k}\dot{W}_3 = 0$$

Therefore,

$$\dot{\overline{W}} = \hat{i}\dot{W}_1 + \hat{j}\dot{W}_2 + \hat{k}\dot{W}_3 \tag{1.47}$$

The center of mass of the gun is located in a Galilean reference frame by vector $\overline{R}_0$, which is given by Eq. (1.2). The external force on the gun is $M\ddot{\overline{R}}_0$. Consequently, by Eqs. (1.38),

$$\overline{F} - Ap_0\hat{v} = -\mu\ddot{\overline{r}} \tag{1.48}$$

where $\mu = Mm/(M + m)$.

A balanced projectile is considered in this article. Consequently, by Eq. (1.13),

$$\overline{r} = \overline{e} + \hat{v}s \tag{1.49}$$

Therefore,

$$\dot{\overline{r}} = \dot{\overline{e}} + \dot{\hat{v}}s + \hat{v}\dot{s}$$

However,

$$\dot{\overline{e}} = \overline{W} \times \overline{e} \text{ and } \dot{\hat{v}} = \overline{W} \times \hat{v}$$

Therefore,

$$\dot{\overline{r}} = \overline{W} \times \overline{r} + \hat{v}\dot{s} \tag{1.50}$$

Differentiation of Eq. (1.50) yields

$$\ddot{\overline{r}} = \dot{\overline{W}} \times \overline{r} + \overline{W} \times \dot{\overline{r}} + \dot{\hat{v}}\dot{s} + \hat{v}\ddot{s}$$

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Hence,

$$\ddot{\overline{r}} = \dot{\overline{W}} \times \overline{r} + \overline{W} \times (\overline{W} \times \overline{r}) + 2\overline{W} \times \hat{\nu}\dot{s} + \hat{\nu}\ddot{s} \qquad (1.51)$$

By expansion of the vector triple product, this becomes

$$\ddot{\overline{r}} = \dot{\overline{W}} \times \overline{r} + \overline{W}\,\overline{W} \cdot \overline{r} - \overline{r}W^2 + 2\overline{W} \times \hat{\nu}\dot{s} + \hat{\nu}\ddot{s} \qquad (1.52)$$

Consequently, by Eq. (1.48),

$$\overline{F} = Ap_0\hat{\nu} - \mu(\dot{\overline{W}} \times \overline{r} + \overline{W}\,\overline{W} \cdot \overline{r} - \overline{r}W^2 + 2\overline{W} \times \hat{\nu}\dot{s} + \hat{\nu}\ddot{s}) \qquad (1.53)$$

With Eq. (1.53), the net force on the gun is determined by Eq. (1.38), and the net force on the projectile is determined by Eq. (1.39). In view of Eq. (1.39), the axial component of force on the projectile is

$$-\overline{F} \cdot \hat{\nu} + Ap_1 \qquad (1.54)$$

For numerical computations, Eq. (1.53) must be expressed in scalar form. The components of $\overline{F}$ are represented by

$$\overline{F} = \hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3$$

Also,

$$\hat{\nu} = \hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma$$

and

$$\overline{r} = \hat{i}r_1 + \hat{j}r_2 + \hat{k}r_3$$

In view of Eq. (1.47),

$$\dot{\overline{W}} \times \overline{r} = \hat{i}(\dot{W}_2 r_3 - \dot{W}_3 r_2) + \hat{j}(\dot{W}_3 r_1 - \dot{W}_1 r_3) + \hat{k}(\dot{W}_1 r_2 - \dot{W}_2 r_1)$$

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Accordingly, Eq. (1.53) yields

$$F_1 = \alpha(Ap_0 - \mu\ddot{s}) - \mu(\dot{W}_2 r_3 - \dot{W}_3 r_2) - \mu W_1(W_2 r_2 + W_3 r_3)$$

$$+ \mu r_1(W_2{}^2 + W_3{}^2) - 2\mu\dot{s}(\gamma W_2 - \beta W_3)$$

$$F_2 = \beta(Ap_0 - \mu\ddot{s}) - \mu(\dot{W}_3 r_1 - \dot{W}_1 r_3) - \mu W_2(W_1 r_1 + W_3 r_3)$$

$$+ \mu r_2(W_3{}^2 + W_1{}^2) - 2\mu\dot{s}(\alpha W_3 - \gamma W_1)$$

$$F_3 = \gamma(Ap_0 - \mu\ddot{s}) - \mu(\dot{W}_1 r_2 - \dot{W}_2 r_1) - \mu W_3(W_1 r_1 + W_2 r_2)$$

$$+ \mu r_3(W_1{}^2 + W_2{}^2) - 2\mu\dot{s}(\beta W_1 - \alpha W_2) \qquad (1.55)$$

The solution of Eq. (1.37) provides the functions W_1, W_2, W_3. Consequently, the force $\overline{F}$ is determined by Eq. (1.55), provided that the center of mass of the projectile lies on the geometric axis of the tube. Then the moment on the projectile is determined by Eq. (1.41), which, in expanded form is

$$M_1' = r_2 F_3 - r_3 F_2 - M_1$$

$$M_2' = r_3 F_1 - r_1 F_3 - M_2$$

$$M_3' = r_1 F_2 - r_2 F_1 - M_3 \qquad (1.56)$$

1.13 FORCE OF AN UNBALANCED PROJECTILE ON THE TUBE

If the projectile is dynamically unbalanced, $\overline{r}$ and $\dot{\overline{r}}$ are given by Eqs. (1.17) and (1.19). The angle χ through which the projectile has turned relative to the tube is

$$\chi = \int_0^t \omega dt \; ; \; \omega = \dot{\chi} \qquad (1.57)$$

Differentiation of Eq. (1.19) yields

$$\ddot{\bar{r}} = \dot{\bar{W}} \times \bar{r} + \bar{W} \times \dot{\bar{r}} + \dot{\hat{v}}\dot{s} + \hat{v}\ddot{s} + \dot{\omega}\hat{v} \times \bar{\epsilon}_0 \cos\chi + \omega\dot{\hat{v}} \times \bar{\epsilon}_0 \cos\chi$$

$$+ \omega\hat{v} \times \dot{\bar{\epsilon}}_0 \cos\chi - \omega^2\hat{v} \times \bar{\epsilon}_0 \sin\chi - \dot{\omega}\bar{\epsilon}_0\sin\chi - \omega\dot{\bar{\epsilon}}_0\sin\chi - \omega^2\bar{\epsilon}_0\cos\chi$$

Now $\dot{\bar{r}}$ can be eliminated by Eq. (1.19). Also,

$$\dot{\hat{v}} = \bar{W} \times \hat{v} \quad \text{and} \quad \dot{\bar{\epsilon}}_0 = \bar{W} \times \bar{\epsilon}_0$$

Consequently,

$$\ddot{\bar{r}} = \dot{\bar{W}} \times \bar{r} + \bar{W} \times [\bar{W} \times \bar{r} + \hat{v}\dot{s} + \omega\hat{v} \times \bar{\epsilon}_0\cos\chi - \omega\bar{\epsilon}_0\sin\chi]$$

$$+ \bar{W} \times \hat{v}\dot{s} + \hat{v}\ddot{s} + \dot{\omega}\hat{v} \times \bar{\epsilon}_0\cos\chi + \omega(\bar{W} \times \hat{v}) \times \bar{\epsilon}_0\cos\chi$$

$$+ \omega\hat{v} \times (\bar{W} \times \bar{\epsilon}_0)\cos\chi - \omega^2\hat{v} \times \bar{\epsilon}_0\sin\chi - \dot{\omega}\bar{\epsilon}_0\sin\chi$$

$$- \omega\bar{W} \times \bar{\epsilon}_0\sin\chi - \omega^2\bar{\epsilon}_0\cos\chi$$

Since

$$(\bar{W} \times \hat{v}) \times \bar{\epsilon}_0 + \hat{v} \times (\bar{W} \times \bar{\epsilon}_0) \equiv \bar{W} \times (\hat{v} \times \bar{\epsilon}_0)$$

this reduces to

$$\ddot{\bar{r}} = \dot{\bar{W}} \times \bar{r} + \bar{W} \times [\bar{W} \times \bar{r} + 2\hat{v}\dot{s} + 2\omega\hat{v} \times \bar{\epsilon}_0\cos\chi - 2\omega\bar{\epsilon}_0\sin\chi]$$

$$+ \hat{v}\ddot{s} + \dot{\omega}(\hat{v} \times \bar{\epsilon}_0\cos\chi - \bar{\epsilon}_0\sin\chi) - \omega^2(\hat{v} \times \bar{\epsilon}_0\sin\chi + \bar{\epsilon}_0\cos\chi)$$

$$(1.58)$$

If $\bar{\epsilon}_0 = 0$, Eq. (1.58) reduces to Eq. (1.52).

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For brevity, vectors $\bar{u}$ and $\bar{v}$ are defined by

$$\hat{v} \times \bar{\epsilon}_0\cos\chi - \bar{\epsilon}_0\sin\chi = \bar{u} = \hat{i}u_1 + \hat{j}u_2 + \hat{k}u_3$$

$$\hat{v} \times \bar{\epsilon}_0\sin\chi + \bar{\epsilon}_0\cos\chi = \bar{v} = \hat{i}v_1 + \hat{j}v_2 + \hat{k}v_3 \qquad (1.59)$$

Also, $\bar{\epsilon}_0 = \hat{i}\epsilon_1 + \hat{j}\epsilon_2 + \hat{k}\epsilon_3$. Consequently,

$$u_1 = (\beta\epsilon_3 - \gamma\epsilon_2)\cos\chi - \epsilon_1\sin\chi$$

$$u_2 = (\gamma\epsilon_1 - \alpha\epsilon_3)\cos\chi - \epsilon_2\sin\chi$$

$$u_3 = (\alpha\epsilon_2 - \beta\epsilon_1)\cos\chi - \epsilon_3\sin\chi \qquad (1.60)$$

$$v_1 = (\beta\epsilon_3 - \gamma\epsilon_2)\sin\chi + \epsilon_1\cos\chi$$

$$v_2 = (\gamma\epsilon_1 - \alpha\epsilon_3)\sin\chi + \epsilon_2\cos\chi$$

$$v_3 = (\alpha\epsilon_2 - \beta\epsilon_1)\sin\chi + \epsilon_3\cos\chi \qquad (1.61)$$

By Eqs. (1.48) and (1.58),

$$\bar{F} = Ap_0\hat{v} - \mu\ddot{\bar{r}}_0 - 2\omega\mu\bar{W} \times \bar{u} - \mu\dot{\omega}\bar{u} + \mu\omega^2\bar{v} \qquad (1.62)$$

where $\bar{r}_0$ is the value of $\bar{r}$ when $\bar{\epsilon}_0 = 0$. Hence, $\ddot{\bar{r}}_0$ is given by Eq. (1.52).
Set

$$\bar{F}' = Ap_0\hat{v} - \mu\ddot{\bar{r}}_0 \qquad (1.63)$$

Then $\bar{F}'$ is the expression that was obtained for $\bar{F}$ in the case of a balanced projectile (Eq. (1.53)). Set

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$$\overline{F} = \overline{F}' + \overline{f} \; ; \; \overline{f} = \hat{i}f_1 + \hat{j}f_2 + \hat{k}f_3 \qquad (1.64)$$

Then $\overline{f}$ is the correction to $\overline{F}'$ to account for unbalance of the projectile. In scalar form, $\overline{F}'$ is given by Eq. (1.55). By Eqs. (1.62), (1.63), and (1.64),

$$\overline{f} = \mu[-2\omega\overline{W} \times \overline{u} - \dot{\omega}\overline{u} + \omega^2\overline{v}] \qquad (1.65)$$

Hence,

$$f_1 = \mu[-2\omega(W_2 u_3 - W_3 u_2) - \dot{\omega}u_1 + \omega^2 v_1]$$

$$f_2 = \mu[-2\omega(W_3 u_1 - W_1 u_3) - \dot{\omega}u_2 + \omega^2 v_2]$$

$$f_3 = \mu[-2\omega(W_1 u_2 - W_2 u_1) - \dot{\omega}u_3 + \omega^2 v_3] \qquad (1.66)$$

The quantities (f_1, f_2, f_3) are the corrections to be added to (F_1, F_2, F_3), respectively, in Eq. (1.55) to account for dynamic unbalance of the projectile. It is to be recalled, however, that $\overline{W}$ is affected to some extent by unbalance of the projectile. Consequently, (W_1, W_2, W_3) are to be computed by Eq. (1.31) rather than by Eq. (1.37). On the other hand, the quantities (W_1, W_2, W_3), determined by Eq. (1.37) are to be used in conjunction with Eq. (1.55). The moment $\overline{M}'$ on the projectile is given by Eq. (1.56) in either case. Equation (1.46), which gives the moment $\overline{M}$ of the contact forces that the projectile exerts on the gun, is valid whether or not the projectile is dynamically balanced. It is to be noted that Eqs. (1.11) and (1.59) show that $\overline{\epsilon} = \overline{v}$.

SECTION 2
FORCES AND MOMENTS ON A RIGID IMMOVABLE GUN
WITH AN UNBALANCED PROJECTILE

2.1   INTRODUCTION

In this section, the gun is considered to be rigid and immovable. The motion of the projectile in the tube is prescribed. The net force and moment acting on the projectile accordingly are determined by the dynamical theory of a single rigid body. The effect of gravity is neglected, but it would merely augment the force $\overline{F}$ by the term $m\overline{g}$, where $\overline{g}$ is the vectorial acceleration of gravity. Gravity would not alter the moment equations.

2.2   KINEMATIC RELATIONS

Since the angular velocity of the gun is zero, the angular velocity of the projectile is

$$\overline{w} = \hat{v}\omega \tag{2.1}$$

where, as before, ω is the spin of the projectile, and $\hat{v}$ is a constant unit vector along the axis of the gun tube.

Equations (1.7), (1.8), (1.9), (1.10), and (1.11) again apply.

Since the origin 0 of coordinates (x,y,z) is now arbitrary, it is conveniently taken to be the initial geometric center Q_0' of the projectile (Figure 1). Then $\overline{R}_0 = \overline{e} = 0$, since the center of mass of the gun is irrelevant. Also, the axes (x,y,z) may be oriented in the directions $\hat{a}_0$, $\hat{b}_0$, $\hat{c}_0$ of the initial principal axes of the projectile. Then $\hat{a}_0 = \hat{i}$, $\hat{b}_0 = \hat{j}$, and $\hat{c}_0 = \hat{k}$. Consequently,

$$\alpha_1 = 1 \; ; \; \beta_1 = 0 \; ; \; \gamma_1 = 0$$

$$\alpha_2 = 0 \; ; \; \beta_2 = 1 \; ; \; \gamma_2 = 0$$

$$\alpha_3 = 0 \; ; \; \beta_3 = 0 \; ; \; \gamma_3 = 1 \tag{2.2}$$

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In view of Eq. (1.8) and Eq. (2.1), $w_1 = \bar{w} \cdot \hat{a} = \omega\hat{v} \cdot \hat{a} = \omega\hat{v} \cdot \hat{a}_0 = \omega\hat{v} \cdot \hat{i}$ $= \omega\alpha$, and likewise for w_2 and w_3. Consequently,

$$w_1 = \omega\alpha \ ; \ w_2 = \omega\beta \ ; \ w_3 = \omega\gamma \tag{2.3}$$

Hence,

$$\dot{w}_1 = \dot{\omega}\alpha \ ; \ \dot{w}_2 = \dot{\omega}\beta \ ; \ \dot{w}_3 = \dot{\omega}\gamma \tag{2.4}$$

Since $\hat{v}$ and $\bar{\epsilon}_0$ are constant vectors and $\omega = \dot{\chi}$, by Eqs. (1.11) and (1.59)

$$\ddot{\bar{\epsilon}} = -\omega^2\bar{\epsilon} + \dot{\omega}\bar{u} \tag{2.5}$$

Since $\bar{R}_0 = \bar{e} = 0$, the location $\bar{R}_1$ of the center of mass of the projectile is (Figure 1)

$$\bar{r} = \bar{R}_1 = \hat{v}s + \bar{\epsilon} \tag{2.6}$$

Consequently, since $\hat{v}$ is a constant vector, the acceleration of the center of mass of the projectile is

$$\ddot{\bar{r}} = \hat{v}\ddot{s} + \ddot{\bar{\epsilon}} \tag{2.7}$$

Equations (2.5) and (2.7) yield

$$\ddot{\bar{r}} = \hat{v}\ddot{s} - \omega^2\bar{\epsilon} + \dot{\omega}\bar{u} \tag{2.8}$$

With Eq. (1.10), Eq. (2.2) yields

$$\hat{a} = \hat{i}\,[\alpha^2 + (\beta^2 + \gamma^2)\cos\chi] + \hat{j}\,[\alpha\beta(1 - \cos\chi) + \gamma\sin\chi]$$

$$+ \hat{k}\,[\gamma\alpha(1 - \cos\chi) - \beta\sin\chi]$$

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$$\hat{b} = \hat{i}\,[\alpha\beta(1 - \cos\chi) - \gamma\sin\chi] + \hat{j}\,[\beta^2 + (\gamma^2 + \alpha^2)\cos\chi]$$

$$+ \hat{k}\,[\beta\gamma(1 - \cos\chi) + \alpha\sin\chi]$$

$$\hat{c} = \hat{i}\,[\gamma\alpha(1 - \cos\chi) + \beta\sin\chi] + \hat{j}\,[\beta\gamma(1 - \cos\chi) - \alpha\sin\chi]$$

$$+ \hat{k}\,[\gamma^2 + (\alpha^2 + \beta^2)\cos\chi] \tag{2.9}$$

Equation (1.11) yields

$$\bar{\epsilon} \times \hat{v} = \hat{v} \times (\bar{\epsilon}_0 \times \hat{v})\sin\chi + \bar{\epsilon}_0 \times \hat{v}\cos\chi$$

Hence,

$$\bar{\epsilon} \times \hat{v} = (\bar{\epsilon}_0 - \hat{v}\hat{v} \cdot \bar{\epsilon}_0)\sin\chi + \bar{\epsilon}_0 \times \hat{v}\cos\chi \tag{2.10}$$

2.3   FORCES ON THE GUN AND ON THE PROJECTILE

The force that the projectile exerts on the gun by direct contact is denoted by $\bar{F}$, as before. The force of contact that the gun exerts on the projectile is $-\bar{F}$. The net force on the projectile is accordingly

$$-\bar{F} + \hat{v}(p_1 A - R) \tag{2.11}$$

where R is the resistance of air ahead of the projectile. In view of Eq. (2.8), Newton's law yields

$$-\bar{F} + \hat{v}(p_1 A - R) = \hat{v}m\ddot{s} - m\omega^2\bar{\epsilon} + m\dot{\omega}\bar{u} \tag{2.12}$$

Hence, by Eqs. (1.11) and (1.59)

$$\bar{F} = \hat{v}(p_1 A - R - m\ddot{s}) + m\omega^2(\hat{v} \times \bar{\epsilon}_0\sin\chi + \bar{\epsilon}_0\cos\chi)$$

$$- m\dot{\omega}(\hat{v} \times \bar{\epsilon}_0\cos\chi - \bar{\epsilon}_0\sin\chi) \tag{2.13}$$

The axial frictional force that the gun exerts on the projectile is in the direction $-\hat{v}$. Its magnitude is $\bar{F} \cdot \hat{v} = F_f$. Since $\hat{v} \cdot \bar{\epsilon}_0 = 0$, Eq. (2.13) yields

$$F_f = p_1 A - R - m\ddot{s} \tag{2.14}$$

By definition,

$$\hat{v} = \hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma \, , \quad \bar{\epsilon}_0 = \hat{i}\epsilon_1 + \hat{j}\epsilon_2 + \hat{k}\epsilon_3 \tag{2.15}$$

Consequently, in scalar form, Eq. (2.13) is

$$F_1 = \alpha(p_1 A - R - m\ddot{s}) + m\omega^2[(\beta\epsilon_3 - \gamma\epsilon_2)\sin\chi + \epsilon_1\cos\chi]$$

$$- m\dot{\omega}[(\beta\epsilon_3 - \gamma\epsilon_2)\cos\chi - \epsilon_1\sin\chi]$$

$$F_2 = \beta(p_1 A - R - m\ddot{s}) + m\omega^2[(\gamma\epsilon_1 - \alpha\epsilon_3)\sin\chi + \epsilon_2\cos\chi]$$

$$- m\dot{\omega}[(\gamma\epsilon_1 - \alpha\epsilon_3)\cos\chi - \epsilon_2\sin\chi]$$

$$F_3 = \gamma(p_1 A - R - m\ddot{s}) + m\omega^2[(\alpha\epsilon_2 - \beta\epsilon_1)\sin\chi + \epsilon_3\cos\chi]$$

$$- m\dot{\omega}[(\alpha\epsilon_2 - \beta\epsilon_1)\cos\chi - \epsilon_3\sin\chi] \tag{2.16}$$

The vectors $\bar{\epsilon}_0$ and $\hat{v} \times \bar{\epsilon}_0$ are perpendicular to the axis of the tube. The second expression in Eq. (2.13) represents centrifugal force. It is zero if the eccentricity $\bar{\epsilon}_0$ is zero.

In addition to the force $\bar{F}$, the gun experiences the breech force $-\hat{v}p_0 A$ from the gases.

2.4 MOMENT ON THE PROJECTILE

In view of Eq. (1.43), the moment about the center of mass of the projectile of all the forces that act on the projectile is $\bar{M}' - \bar{\epsilon} \times \hat{v}(p_1 A - R)$. (In Eq. (1.43), R was neglected.) The components of this moment on the principal axes of the inertia of the projectile are obtained by taking the

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scalar products of the moment with $\hat{a}$, $\hat{b}$, and $\hat{c}$. By virtue of Eqs. (2.3) and (2.4), Euler's equations (Ref. 3) yield

$$\bar{M}' \cdot \hat{a} = (p_1 A - R)\bar{\epsilon} \times \hat{\nu} \cdot \hat{a} + i_1 \alpha\dot{\omega} - (i_2 - i_3)\beta\gamma\omega^2$$

$$\bar{M}' \cdot \hat{b} = (p_1 A - R)\bar{\epsilon} \times \hat{\nu} \cdot \hat{b} + i_2 \beta\dot{\omega} - (i_3 - i_1)\gamma\alpha\omega^2$$

$$\bar{M}' \cdot \hat{c} = (p_1 A - R)\bar{\epsilon} \times \hat{\nu} \cdot \hat{c} + i_3 \gamma\dot{\omega} - (i_1 - i_2)\alpha\beta\omega^2 \qquad (2.17)$$

If $\bar{U}$ is any vector, $\hat{a}\hat{a} \cdot \bar{U} + \hat{b}\hat{b} \cdot \bar{U} + \hat{c}\hat{c} \cdot \bar{U} \equiv \bar{U}$. Consequently, Eq. (2.17) yields

$$\bar{M}' = (p_1 A - R)\bar{\epsilon} \times \hat{\nu} + \dot{\omega}(i_1 \alpha\hat{a} + i_2 \beta\hat{b} + i_3 \gamma\hat{c}) - \omega^2[(i_2 - i_3)\beta\gamma\hat{a}$$

$$+ (i_3 - i_1)\gamma\alpha\hat{b} + (i_1 - i_2)\alpha\beta\hat{c}] \qquad (2.18)$$

The constant vectors $\hat{\nu} = \hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma$ and $\bar{\epsilon}_0 = \hat{i}\epsilon_1 + \hat{j}\epsilon_2 + \hat{k}\epsilon_3$ are considered to be known. Also, $\chi(t)$, $\omega = \dot{\chi}$, $p_1(t)$ and $R(t)$ are regarded as known functions. Accordingly, in view of Eqs. (1.11) and (2.9), $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\bar{\epsilon}$ are known vector functions of t. Consequently, Eq. (2.18) is an explicit vector formula for the moment $\bar{M}'(t)$. Equation (2.10) is a representation of the vector product $\bar{\epsilon} \times \hat{\nu}$ that occurs in Eq. (2.18). Accordingly, the vector $\bar{M}'$, defined by Eq. (2.18), may be resolved into its $\hat{i}$, $\hat{j}$, $\hat{k}$ components. Setting $\bar{M}' = \hat{i}M_1' + \hat{j}M_2' + \hat{k}M_3'$, we get

$$M_1' = (p_1 A - R)[\epsilon_1 \sin\chi - \alpha(\alpha\epsilon_1 + \beta\epsilon_2 + \gamma\epsilon_3)\sin\chi$$

$$+ (\gamma\epsilon_2 - \beta\epsilon_3)\cos\chi] + [\dot{\omega}i_1\alpha - \omega^2(i_2 - i_3)\beta\gamma][\alpha^2$$

$$+ (\beta^2 + \gamma^2)\cos\chi] + [\dot{\omega}i_2\beta - \omega^2(i_3 - i_1)\gamma\alpha][\alpha\beta\text{vers}\chi$$

$$- \gamma\sin\chi] + [\dot{\omega}i_3\gamma - \omega^2(i_1 - i_2)\alpha\beta][\gamma\alpha\text{vers}\chi + \beta\sin\chi]$$

$$M_2' = (p_1 A - R)[\varepsilon_2 \sin\chi - \beta(\alpha\varepsilon_1 + \beta\varepsilon_2 + \gamma\varepsilon_3)\sin\chi + (\alpha\varepsilon_3 - \gamma\varepsilon_1)\cos\chi]$$

$$+ [\dot{\omega}i_1\alpha - \omega^2(i_2 - i_3)\beta\gamma][\alpha\beta\text{vers}\chi + \gamma\sin\chi] + [\dot{\omega}i_2\beta$$

$$- \omega^2(i_3 - i_1)\gamma\alpha][\beta^2 + (\gamma^2 + \alpha^2)\cos\chi] + [\dot{\omega}i_3\gamma$$

$$- \omega^2(i_1 - i_2)\alpha\beta][\beta\gamma\text{vers}\chi - \alpha\sin\chi]$$

$$M_3' = (p_1 A - R)[\varepsilon_3 \sin\chi - \gamma(\alpha\varepsilon_1 + \beta\varepsilon_2 + \gamma\varepsilon_3)\sin\chi + (\beta\varepsilon_1 - \alpha\varepsilon_2)\cos\chi]$$

$$+ [\dot{\omega}i_1\alpha - \omega^2(i_2 - i_3)\beta\gamma][\gamma\alpha\text{vers}\chi - \beta\sin\chi] + [\dot{\omega}i_2\beta$$

$$- \omega^2(i_3 - i_1)\gamma\alpha][\beta\gamma\text{vers}\chi + \alpha\sin\chi] + [\dot{\omega}i_3\gamma - \omega^2(i_1 - i_2)\alpha\beta]$$

$$[\gamma^2 + (\alpha^2 + \beta^2)\cos\chi] \tag{2.19}$$

where $\text{vers}\chi = 1 - \cos\chi$.

    The moment $\overline{M}$ of the contact forces that the projectile exerts on the gun about the origin Q_0' is given by Eq. (1.41); namely,

$$\overline{M} = \overline{r} \times \overline{F} - \overline{M}' \tag{2.20}$$

where $\overline{r} = \hat{v}s + \overline{\varepsilon}$.

    If the projectile is balanced, $\overline{\varepsilon}_0 = 0$ and $\hat{v}$ lies along a principal axis of inertia of the projectile - say $\hat{v} = \hat{k}$. Then Eq. (2.19) yields $M_1' = M_2' = 0$ and $M_3' = i_3\dot{\omega}$. In this case, M_3' is the rifling torque. For an unbalanced projectile, the rifling torque may be defined as $\overline{M}' \cdot \hat{v}$, where $\overline{M}'$ is represented by Eq. (2.18). The term $(p_1 A - R)\overline{\varepsilon} \times \hat{v}$ cancels out of the scalar product $\overline{M}' \cdot \hat{v}$.

SECTION 3

FORCES AND MOMENTS ON A FREELY-RECOILING
RIGID GUN THAT IS CONSTRAINED AGAINST ROTATION

3.1   INTRODUCTION

A rigid gun translates freely along an inclined guide, represented by an inclined plane in Figure 2. It is constrained against rotation. The angle θ of the inclined plane may differ from the angle of elevation $\theta + \phi$ of the barrel. Air resistance to motion of the projectile and effects of gravity are included in the analysis. It is questionable whether gravity is meaningful in this problem, since it would cause the system to slide down the inclined plane with increasing speed. It is easily eliminated, however, by setting $g = 0$.

In addition to notations introduced previously, a few new notations are added.

$\hat{\mu}$ is a unit vector along the axis of the recoil guide (Figure 2).

ξ, η are rectangular coordinates fixed in a Galilean reference frame that contains the recoil guide (Figure 2).

u is the recoil displacement along the guide (Figure 2).

$\overline{R}_1$ is the vector from the origin of the coordinates (ξ, η) to the center of mass of the projectile (Figure 2).

3.2   MOTION OF THE SYSTEM

The velocity of the projectile relative to the gun is $\dot{s}$. The component of this velocity along the guide is $\dot{s}\cos\phi$, as is seen by Figure 2. Consequently, the momentum of the system in the direction of the guide is

$$m(\dot{s}\cos\phi - \dot{u}) - M\dot{u} = m\dot{s}\cos\phi - (M + m)\dot{u}$$

The component of external force on the system in the direction of the guide is

$$-(M + m)g\sin\theta + (Ap_1 - Ap_0 - R)\cos\phi$$

where R is the resisting force of air in the tube ahead of the projectile. Since the external force is equal to the rate of change of momentum,

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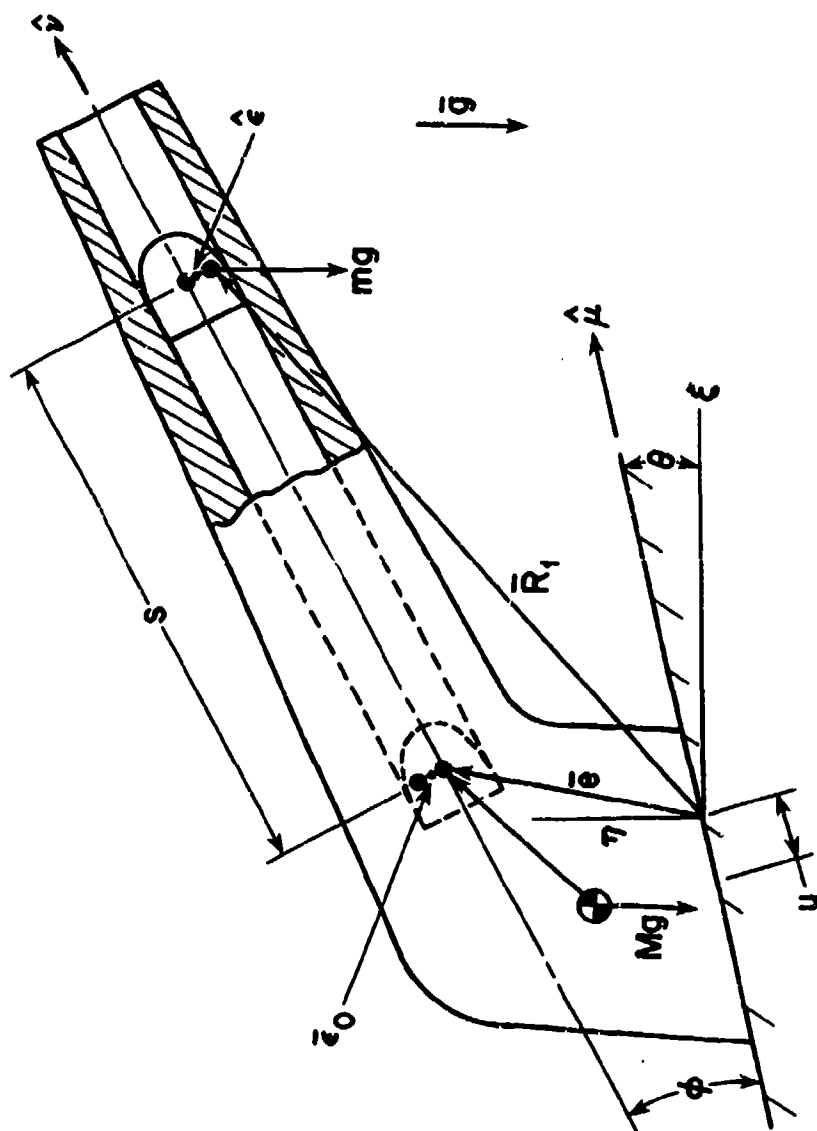
Figure 2. Gun Schematic and Notations

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$$m\ddot{s}\cos\phi - (M + m)\ddot{u} = -(M + m)g\sin\theta + (Ap_1 - Ap_0 - R)\cos\phi$$

or

$$\ddot{u} = g\sin\theta + \frac{m}{M + m} \ddot{s}\cos\phi - \frac{(Ap_1 - Ap_0 - R)}{M + m} \cos\phi \qquad (3.1)$$

Since the gun is constrained against rotation, the absolute angular velocity of the projectile is

$$\overline{w} = \hat{\nu}\omega \qquad (3.2)$$

Since ω, s, and R are regarded as known functions of t, Eqs. (3.1) and (3.2) determine the motion of the gun and the projectile.

3.3 FORCES IN THE SYSTEM

The resultant force of contact that the projectile exerts on the tube is $\overline{F}$. The resultant force of contact that the tube exerts on the projectile is $-\overline{F}$. If the gun is regarded as a free body, the net force on it is

$$\overline{F} - Ap_0\hat{\nu} + M\overline{g} \qquad (3.3)$$

where p_0 is the gas pressure at the breech. The net force on the projectile is

$$-\overline{F} + m\overline{g} + (Ap_1 - R)\hat{\nu} \qquad (3.4)$$

where p_1 is the gas pressure on the base of the projectile. Consequently, by Newton's law

$$-\overline{F} + m\overline{g} + (Ap_1 - R)\hat{\nu} = m\ddot{\overline{R}}_1 \qquad (3.5)$$

By Figure 2,

$$\overline{R}_1 = \overline{e} + \hat{\nu}s + \overline{\epsilon} \qquad (3.6)$$

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Also,

$$\bar{e} = \bar{e}_0 - \hat{\mu}u \qquad (3.7)$$

where $\bar{e}_0$ is the initial value of $\bar{e}$. Hence,

$$\bar{R}_1 = \bar{e}_0 - \hat{\mu}u + \hat{\nu}s + \bar{\epsilon} \qquad (3.8)$$

Therefore,

$$\ddot{\bar{R}}_1 = -\hat{\mu}\ddot{u} + \hat{\nu}\ddot{s} + \ddot{\bar{\epsilon}} \qquad (3.9)$$

Equations (1.11) and (2.5) are again applicable. Equations (3.1), (3.5), (3.9), (1.11), (1.59), and (2.5) yield

$$\bar{F} = m\bar{g} + (Ap_1 - R)\hat{\nu} + \hat{\mu}mg\sin\theta + \frac{\hat{\mu}m^2}{M + m}\ddot{s}\cos\phi - \frac{\hat{\mu}m}{M + m}(Ap_1 - Ap_0$$
$$- R)\cos\phi - m\hat{\nu}\ddot{s} + m\omega^2(\hat{\nu} \times \bar{\epsilon}_0\sin\chi + \bar{\epsilon}_0\cos\chi) - m\dot{\omega}(\hat{\nu} \times \bar{\epsilon}_0\cos\chi$$
$$- \bar{\epsilon}_0\sin\chi) \qquad (3.10)$$

The magnitude of the axial friction force on the projectile is $\bar{F} \cdot \hat{\nu} = F_f$. Hence,

$$F_f = -mg\cos\theta\sin\phi + (Ap_1 - R - m\ddot{s}) \frac{M + m\sin^2\phi}{M + m} + \frac{mAp_0}{M + m} \cos^2\phi \qquad (3.11)$$

If $M \to \infty$, F_f reduces to Eq. (2.14), aside from the g-term which was neglected in the derivation of Eq. (2.14).

3.4   MOMENT ON THE PROJECTILE

Since the motion of the projectile relative to the gun is presumed to be prescribed, the recoil of the gun merely superimposes a translation on the absolute motion of the projectile. The recoil has no effect on the angular velocity of the projectile. Therefore, in view of Euler's equations (Ref. 3), it has no effect on the resultant moment about the center of mass of the projectile. Likewise, gravity has no effect on

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this moment. The theory of moments on a projectile in a motionless rigid gun consequently is directly applicable to the recoiling gun. The moment $\overline{M}'$ about the center of mass of the projectile is again given by Eq. (2.18). The components of $\overline{M}'$ are again given by Eq. (2.19).

SECTION 4
RECOILING RIGID GUN WITH OFFSET BREECH
AND FIXED TRUNNION

4.1   INTRODUCTION

If the center of mass of the breech of a gun lies below the axis of the tube, the recoil causes the muzzle to jerk upward when the gun is fired. Because of its spin, the projectile then exerts a gyroscopic couple that tends to turn the tube sideways. It is assumed in this section that a constraint is provided which prevents rotation of the gun about a vertical axis. Then the spin of the projectile has no effect on the motion of the gun. Also, because of this constraint, offsetting of the breech block to the right or the left has no kinematic effect. When the gun is fired, each particle describes a curve that lies in a plane perpendicular to the axis of the trunnion. Because of the constraint provided by the trunnion, the gun has only two degrees of freedom. The projectile adds another degree of freedom to the system.

Figure 3 is a schematic side view of the gun. The trunnion is fixed in a Galilean reference frame; e.g., the earth. The recoil mechanism is represented schematically as a spring in a slot. One end of the spring is attached to the breech block, and the other end to the trunnion. The slot slides freely over the fixed trunnion. The spring need not be Hookean. Rather, the force F exerted by the recoil mechanism is regarded as an unspecified function of u and $\dot{u}$, where u is the displacement of the breech along the axis of the slot (Figure 3). Accordingly, the recoil mechanism may contain nonlinear springs and nonlinear dashpots. Also, Coulomb friction is admitted. For generality, the line of action of the recoil mechanism is not taken parallel to the axis of the tube.

The projectile is considered to be a body of revolution with its center of mass on its axis of symmetry, and with one principal axis of inertia coinciding with the axis of symmetry i.e., the projectile is perfectly balanced.

4.2   NOTATIONS

Some deviations from previous notations are necessary. Also, a few notations are added.
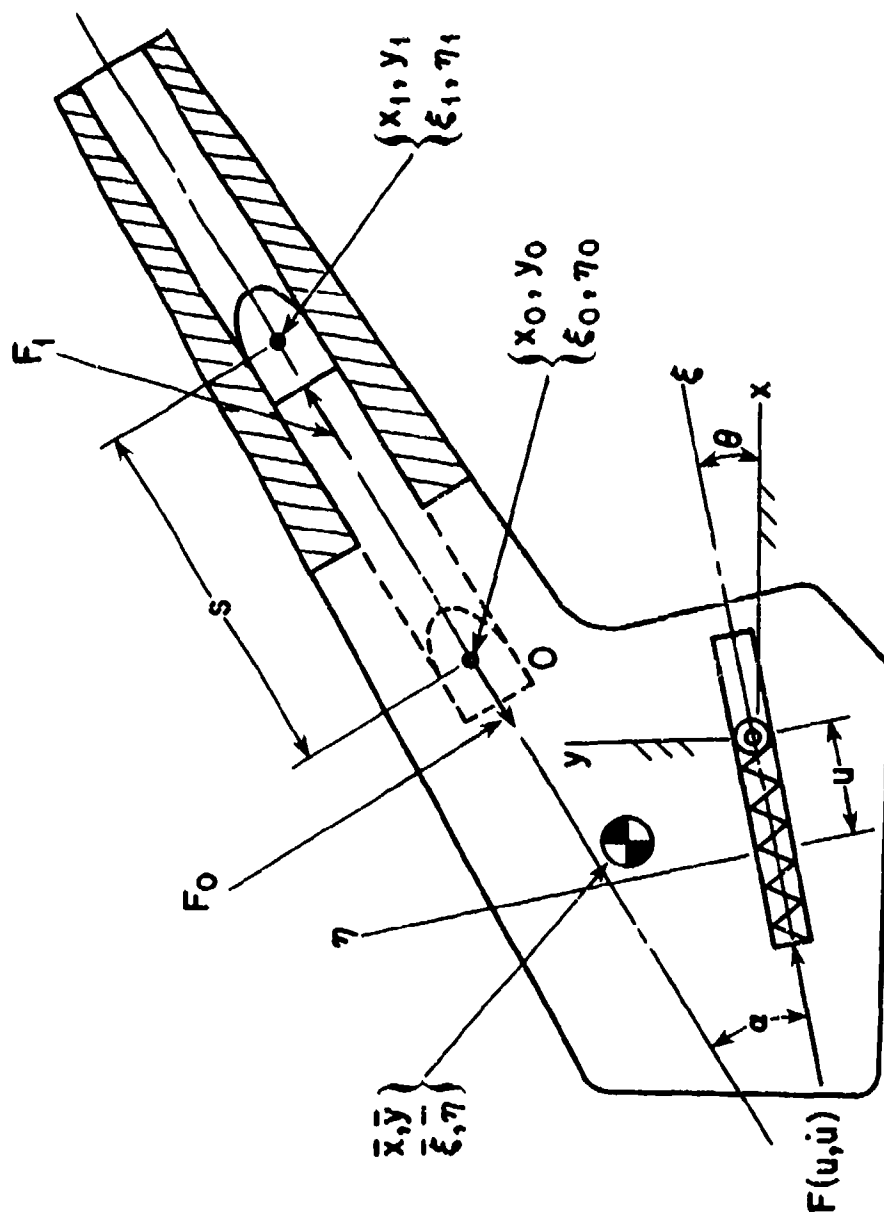
Figure 3. Notations

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x,y are rectangular coordinates with the y-axis vertical, and the origin at the trunnion. They are fixed in a Galilean reference frame (Figure 3).

α is the angle between the axis of the tube and the axis of the recoil mechanism. It is a constant (Figure 3).

u is the displacement of the gun along the axis of the recoil mechanism (Figure 3).

ξ, η are rectangular coordinates scribed on the breech. The ξ-axis is the axis of the recoil mechanism.

θ is the angle between the ξ and x axes. Generalized coordinates are θ, u.

ξ_0, η_0 are the ξ, η coordinates of the center of mass of the projectile before firing.

$\bar{\xi}, \bar{\eta}$ are the ξ, η coordinates of the center of mass of the gun.

$\bar{x}, \bar{y}$ are the x, y coordinates of the center of mass of the gun.

x_0, y_0 are the x, y coordinates of the center of mass of the projectile before firing.

ξ_1, η_1 are the ξ, η coordinates of the center of mass of the projectile at time t.

x_1, y_1 are the x, y coordinates of the center of mass of the projectile at time t.

F_0 is the force of gas pressure on the base of the tube at time t.

F_1 is the net force driving the projectile at time t (force of gas pressure minus the friction of the barrel). F_1 includes the resistance of air ahead of the projectile.

$F(u, \dot{u})$ is the force that the recoil mechanism exerts on the breech.

s is the distance that the projectile has traveled relative to the gun at time t.

M is the mass of the gun.

m is the mass of the projectile.

I is the moment of inertia of the gun about a transverse axis through the center of mass of the gun.

i is the moment of inertia of the projectile about a transverse axis through the center of mass of the projectile.

T is the kinetic energy of the system.

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g is the acceleration of gravity.

Q_1, Q_2, Q_3 are components of generalized force. $\delta W = Q_1 \delta u + Q_2 \delta \theta + Q_3 \delta s$.

4.3 COORDINATE TRANSFORMATION

By Figure 4,

$$x = (\xi - u)\cos\theta - \eta\sin\theta$$

$$y = (\xi - u)\sin\theta + \eta\cos\theta \tag{4.1}$$

Consequently,

$$\bar{x} = (\bar{\xi} - u)\cos\theta - \bar{\eta}\sin\theta$$

$$\bar{y} = (\bar{\xi} - u)\sin\theta + \bar{\eta}\cos\theta \tag{4.2}$$

$$x_0 = (\xi_0 - u)\cos\theta - \eta_0\sin\theta$$

$$y_0 = (\xi_0 - u)\sin\theta + \eta_0\cos\theta \tag{4.3}$$

$$x_1 = (\xi_1 - u)\cos\theta - \eta_1\sin\theta$$

$$y_1 = (\xi_1 - u)\sin\theta + \eta_1\cos\theta \tag{4.4}$$

Also, by Figure 3,

$$\xi_1 = \xi_0 + s\cos\alpha$$

$$\eta_1 = \eta_0 + s\sin\alpha \tag{4.5}$$

Therefore,

$$x_1 = (\xi_0 - u)\cos\theta - \eta_0\sin\theta + s\cos(\theta + \alpha)$$
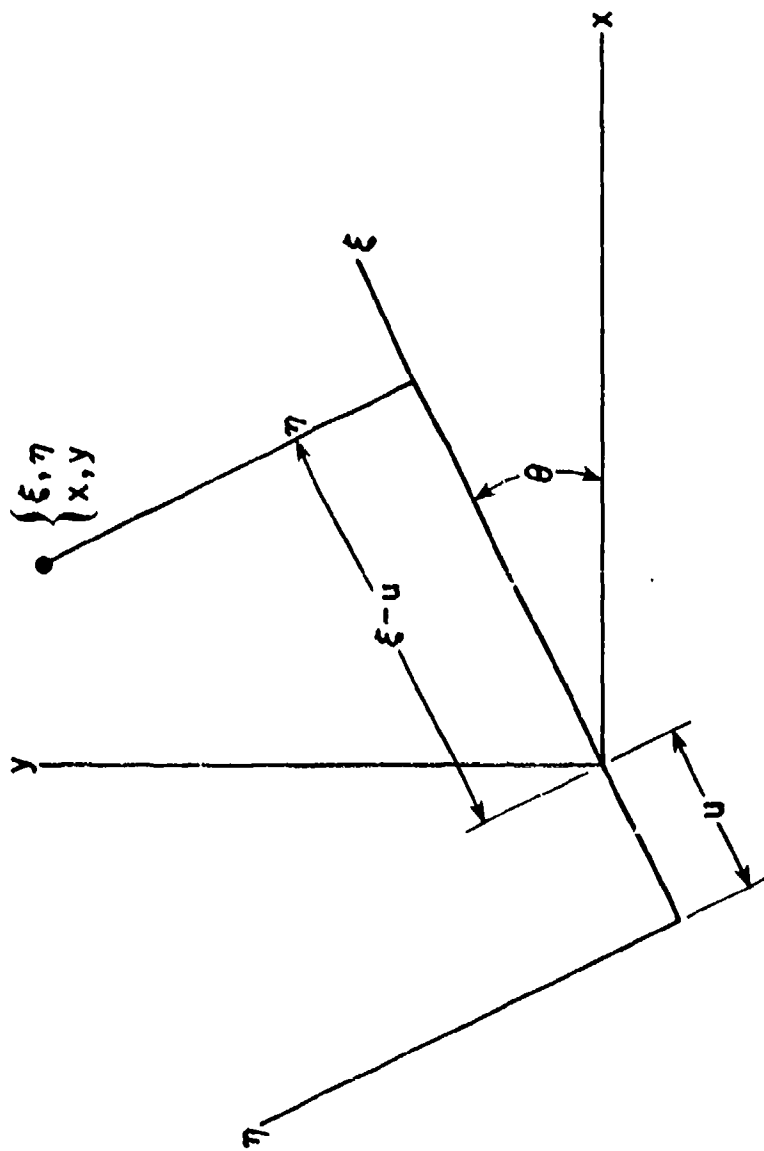
Figure 4. Coordinate Transformation

$$y_1 = (\xi_0 - u)\sin\theta + \eta_0\cos\theta + s\sin(\theta + \alpha) \tag{4.6}$$

By Eq. (4.2),

$$\dot{\bar{x}}^2 + \dot{\bar{y}}^2 = (\bar{\xi} - u)^2\dot{\theta}^2 + (\dot{u} + \bar{\eta}\dot{\theta})^2 \tag{4.7}$$

By Eq. (4.4),

$$\dot{x}_1^2 + \dot{y}_1^2 = [\dot{\theta}(\xi_1 - u) + \dot{\eta}_1]^2 + [\dot{\xi}_1 - \dot{u} - \eta_1\dot{\theta}]^2 \tag{4.8}$$

By Eq. (4.5),

$$\dot{\xi}_1 = \dot{s}\cos\alpha \; ; \; \dot{\eta}_1 = \dot{s}\sin\alpha \tag{4.9}$$

4.4   KINETIC ENERGY

The kinetic energy of the gun is

$$T_g = \frac{1}{2}\,I\dot{\theta}^2 + \frac{1}{2}\,M(\dot{\bar{x}}^2 + \dot{\bar{y}}^2)$$

The kinetic energy of the projectile is

$$T_p = \frac{1}{2}\,i\dot{\theta}^2 + \frac{1}{2}\,m(\dot{x}_1^2 + \dot{y}_1^2)$$

Consequently, in view of Eqs. (4.5), (4.7), (4.8), and (4.9), the kinetic energy of the system is

$$T = \frac{1}{2}(I + i)\dot{\theta}^2 + \frac{1}{2}\,M[(\bar{\xi} - u)^2\dot{\theta}^2 + (\dot{u} + \bar{\eta}\dot{\theta})^2]$$

$$+ \frac{1}{2}\,m[\dot{\theta}(\xi_0 + s\cos\alpha - u) + \dot{s}\sin\alpha]^2$$

$$+ \frac{1}{2}\,m[\dot{s}\cos\alpha - \dot{u} - \dot{\theta}(\eta_0 + s\sin\alpha)]^2 \tag{4.10}$$

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4.5   GENERALIZED FORCE

If the coordinates (u, θ, s) receive virtual increments $(\delta u, \delta \theta, \delta s)$, the work of all the forces that act on the system is a linear form in these increments; i.e.,

$$\delta W = Q_1 \delta u + Q_2 \delta \theta + Q_3 \delta s \qquad (4.11)$$

Q_1, Q_2, Q_3 are called "components of generalized force."

The virtual work of gravity is

$$\delta W_{gr} = -Mg\delta\bar{y} - mg\delta y_1$$

By Eq. (4.2),

$$\delta\bar{y} = -(\sin\theta)\delta u + [(\bar{\xi} - u)\cos\theta]\delta\theta - (\bar{\eta}\sin\theta)\delta\theta$$

By Eq. (4.6),

$$\delta y_1 = [(\xi_0 - u)\cos\theta - \eta_0\sin\theta + s\cos(\theta + \alpha)]\delta\theta - \delta u \sin\theta$$

$$+ \sin(\theta + \alpha)\delta s$$

Consequently,

$$\delta W_{gr} = (M + m)g\sin\theta\, \delta u - Mg[(\bar{\xi} - u)\cos\theta - \bar{\eta}\sin\theta]\delta\theta$$

$$- mg[(\xi_0 - u)\cos\theta - \eta_0\sin\theta]\delta\theta - mgs\cos(\theta + \alpha)\delta\theta$$

$$- mg\sin(\theta + \alpha)\delta s \qquad (4.12)$$

The virtual work of gas pressure on the breech (see Figure 3) is

$$\delta W_{br} = -F_0\cos(\theta + \alpha)\delta x_0 - F_0\sin(\theta + \alpha)\delta y_0$$

Consequently, by Eq. (4.3),

$$\delta W_{br} = F_0 \cos\alpha \delta u - F_0 [(\xi_0 - u)\sin\alpha - \eta_0 \cos\alpha]\delta\theta \tag{4.13}$$

The virtual work that the recoil mechanism performs on the breech is

$$\delta W_r = -F\delta u \tag{4.14}$$

The component of absolute virtual displacement of the projectile along the axis of the tube is

$$\delta x_1 \cos(\theta + \alpha) + \delta y_1 \sin(\theta + \alpha)$$

With Eq. (4.6), this reduces to

$$\delta s - \delta u \cos\alpha + (\xi_0 - u)\sin\alpha\delta\theta - \eta_0 \cos\alpha\delta\theta$$

Consequently, the virtual work performed on the projectile is

$$\delta W_{pr} = F_1 \delta s - F_1 \delta u \cos\alpha + F_1 (\xi_0 - u)\sin\alpha\delta\theta - F_1 \eta_0 \cos\alpha\delta\theta \tag{4.15}$$

Equations (4.11) to (4.15) yield

$$Q_1 = (M + m)g\sin\theta + (F_0 - F_1)\cos\alpha - F(u,\dot{u})$$

$$Q_2 = -Mg[(\overline{\xi} - u)\cos\theta - \overline{\eta}\sin\theta] - mg[(\xi_0 - u)\cos\theta - \eta_0 \sin\theta]$$

$$- mgs\cos(\theta + \alpha) - (F_0 - F_1)[(\xi_0 - u)\sin\alpha - \eta_0 \cos\alpha]$$

$$Q_3 = -mg\sin(\theta + \alpha) + F_1 \tag{4.16}$$

The component Q_2 may be identified as the counterclockwise moment about the trunnion of all external forces acting on the system. The forces F_0 and F_1 are regarded as external forces.

4.6 LAGRANGE'S EQUATIONS

The Lagrange equations are

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - \frac{\partial T}{\partial u} = Q_1$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}}) = Q_2$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{s}}) - \frac{\partial T}{\partial s} = Q_3 \tag{4.17}$$

The term $\partial T/\partial \theta$ is missing because T does not depend on θ (see Eq. (4.10)). The derivative $\partial T/\partial \dot{\theta}$ may be identified as the angular momentum of the system about the trunnion. Consequently, the second Lagrange equation expresses the fact that the moment of all the forces about the trunnion equals the rate of increase of the angular momentum of the system about the trunnion. It might be argued that F_0 and F_1 are internal forces, and, in the absence of gravity, the angular momentum of the system about the trunnion is constant. However, this is true only if the gases in the tube are included in the system. Consequently, it is best to regard F_0 and F_1 as external forces. Force F_1 may include the resistance of air in the tube ahead of the projectile, which is clearly an external force.

With Eq. (4.16), the first Lagrange equation yields,

$$(M + m)\ddot{u} + (M\overline{\eta} + m\eta_G + ms\sin\alpha)\ddot{\theta} - m\ddot{s}\cos\alpha - (M + m)u\dot{\theta}^2$$

$$+ (M\overline{\xi} + m\xi_0 + ms\cos\alpha)\dot{\theta}^2 + 2m\dot{s}\dot{\theta}\sin\alpha + F(u,\dot{u})$$

$$= (M + m)g\sin\theta + (F_0 - F_1)\cos\alpha \tag{4.18}$$

By Eq. (4.10),

$$\frac{\partial T}{\partial \dot{\theta}} = [M\bar{\eta} + m(\eta_0 + s\sin\alpha)]\dot{u} + [I + i + M(\bar{\xi} - u)^2 + M\bar{\eta}^2$$

$$+ m(\xi_0 - u + s\cos\alpha)^2 + m(\eta_0 + s\sin\alpha)^2]\dot{\theta}$$

$$+ m[(\xi_0 - u)\sin\alpha - \eta_0\cos\alpha]\dot{s} \qquad (4.19)$$

Accordingly, by Eqs. (4.16) and (4.17), the second Lagrange equation is

$$\frac{d}{dt}\{[M\bar{\eta} + m(\eta_0 + s\sin\alpha)]\dot{u} + [I + i + M(\bar{\xi} - u)^2 + M\bar{\eta}^2$$

$$+ m(\xi_0 - u + s\cos\alpha)^2 + m(\eta_0 + s\sin\alpha)^2]\dot{\theta} + m[(\xi_0 - u)\sin\alpha$$

$$- \eta_0\cos\alpha]\dot{s}\} = - Mg[(\bar{\xi} - u)\cos\theta - \bar{\eta}\sin\theta] - mg[(\xi_0 - u)\cos\theta$$

$$- \eta_0\sin\theta] - mgs\cos(\theta + \alpha) - (F_0 - F_1)\cdot$$

$$[(\xi_0 - u)\sin\alpha - \eta_0\cos\alpha] \qquad (4.20)$$

The Lagrange equation corresponding to s is

$$[(\xi_0 - u)\sin\alpha - \eta_0\cos\alpha]\ddot{\theta} + \ddot{s} - \ddot{u}\cos\alpha - \dot{\theta}^2(\xi_0 - u)\cos\alpha$$

$$- \dot{\theta}^2 s - 2\dot{u}\dot{\theta}\sin\alpha - \dot{\theta}^2\eta_0\sin\alpha = -g\sin(\theta + \alpha) + \frac{F_1}{m} \qquad (4.21)$$

Equations (4.18) to (4.21) are simplified considerably if $\alpha = 0$.

The displacement s of the projectile and the base pressure force F_0 may be regarded as known functions of t. Equation (4.21) may be used to eliminate F_1 from Eqs. (4.18) and (4.20). After F_1 is eliminated, Eqs. (4.18) and (4.20) are two nonlinear coupled second-order ordinary

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differential equations that determine the functions $u(t)$ and $\theta(t)$, if initial values u_0, $\dot{u}_0$, θ_0, $\dot{\theta}_0$ are given. After $u(t)$ and $\theta(t)$ are determined, $F_1(t)$ can be calculated by Eq. (4.21).

Rotary friction and rotary spring resistance in the trunnion have been disregarded, but their inclusion in the equations is simple. The right side of Eq. (4.20) is merely augmented by a term $-\phi(\theta,\dot{\theta})$, which represents the resisting moment of the trunnion.

If the system starts from rest, the quantities u, $\dot{u}$, $\dot{\theta}$ are small while the projectile remains in the barrel. Consequently, it is reasonable to linearize the differential equations in these variables. At least, this approximation provides a start for an iterative solution of the nonlinear equations.

SECTION 5

CONCLUSIONS

The equations are rather complicated, but they appear to be amendable to numerical treatment with a digital computer. No numerical results are included in this report.

Section 1 treats a gun that is unsupported. In an actual gun, the effect of the recoil mechanism may be negligible during the few milliseconds that the moving projectile remains in the barrel. The trunnion provides a constraint, but it may be temporarily ineffective if there are appreciable clearances in the bearings. Consequently, the unsupported gun might not deviate unduly from reality in some cases.

Equations (1.31) are the key equations in Section 1. For a balanced projectile, they reduce to Eq. (1.37). The unknowns in Eq. (1.31) or (1.37) are the angular velocity components (W_1, W_2, W_3). Equations (1.31) or (1.37) are linear algebraic equations in these variables. Consequently, they are immediately solvable. After (W_1, W_2, W_3) are calculated, the direction cosines (ℓ_i, m_i, n_i) of the principal axes of inertia of the gun are determined as functions of time t by solving the nine linear first-order differential equations (1.33) or, alternatively, by solving the three nonlinear first-order differential equations (1.36). Apparently, these equations must be solved numerically. After (ℓ_i, m_i, n_i) are determined, the motion of the center of mass is determined by Eq. (1.4). Thus, the motion of the system is determined completely, since the motion of the projectile relative to the gun is presumed to be known. Since the angular velocity components (W_1, W_2, W_3) have been calculated, the components (M_1, M_2, M_3) of the moment $\overline{M}$ (exerted about the center of mass of the gun by the contact forces of the projectile) are determined by Euler's dynamical equations for a rigid body (Eq. (1.46)). The force $\overline{F}$ that a balanced projectile exerts on the tube is determined by Eq. (1.55). For an unbalanced projectile, this force must be augmented by the corrective terms in Eq. (1.66). The rifling torque and the axial friction force on the projectile can be calculated directly after the vectors $\overline{M}$ and $\overline{F}$ are determined.

Section 2 treats a gun that is immovable. The force $\overline{F}$ that the projectile exerts on the gun is given, in this case, by Eq. (2.13), or, in scalar form, by Eq. (2.16). The moment $\overline{M}'$ about the center of mass of the projectile of the contact forces imposed by the tube is given by Eq. (2.18), or, in scalar form, by Eq. (2.19). These equations are explicit algebraic formulas for $\overline{F}$ and $\overline{M}'$. Other pertinent forces and moments are immediately determinate when $\overline{F}$ and $\overline{M}'$ are known.

Section 3 treats a gun that translates freely along a guide, but it is constrained against rotation. In this case, the moment $\overline{M}'$ is the same as for the fixed gun, treated in Section 2. The recoil displacement u is determined by integrating Eq. (3.1). The force of contact $\overline{F}$ that the projectile exerts on the tube is given by Eq. (3.10). The magnitude of the axial frictional force on the projectile is given by Eq. (3.11).

Section 4 treats a gun with a trunnion and a general type of recoil device (see Figure 3). The system has two degrees of freedom, corresponding to the recoil displacement u and the angular displacement θ of the gun. A third coordinate s is introduced. It represents the axial displacement of the projectile with respect to the gun, but, since this is presumed to be given, the Lagrange equation corresponding to s merely determines the net driving force F_1 on the projectile. After F_1 is eliminated, Eqs. (4.18) and (4.20) are two nonlinear second-order ordinary differential equations that determine the functions u(t) and $\theta(t)$, if initial values u_0, $\dot{u}_0$, θ_0, $\dot{\theta}_0$ are given. A numerical program to carry out this solution is needed.

Within the frameworks of the mathematical models that are used, the analyses are exact. However, the problem of gas dynamics in the tube is not rigorously separable from the problem of dynamical response of the gun. The effects of momentum and kinetic energy of the charge requires further study.

Finally, it may be advisable to issue as separate complete working reports the results of Sections 1, 2, 3, and 4.

ACKNOWLEDGEMENTS

REFERENCES

1. R. Courant and K. Friedrichs, "Supersonic Flow and Shock Waves (U)," Chap. III, Interscience Publishers, New York, 1948.

2. H. L. Langhaar and A. P. Boresi, Engineering Mechanics-Dynamics (U), McGraw-Hill, New York, 1959, Art. 15-10.

3. H. L. Langhaar, Energy Methods in Applied Mechanics (U), John Wiley and Sons, New York, 1962, Chap. 7.

4. E. L. Ince, Ordinary Differential Equations (U), Dover Pubs., New York, 1944, Arts. 3.3 and 3.31.

APPENDIX A

DISPLACEMENT VECTOR FIELD OF A RIGID BODY

This appendix presents a derivation of the displacement vector field of a rigid body that undergoes an angular displacement χ about an axis A with direction $\hat{v}$ relative to given rectangular coordinates (ξ, η, ζ). The axis A is conveniently taken to pass through the origin O (Figure A-1).

A particle P of the body describes a circular arc C of radius a, whose plane is perpendicular to axis A, and whose center M is on axis A. The displacement vector $\bar{q}$ of particle P may be resolved into components $\bar{q}_1$ and $\bar{q}_2$, such that $\bar{q}_1$ is tangent to circle C at the initial point P, and $\bar{q}_2$ is parallel to the radius MP. The radius vector $\overrightarrow{OP}$ is denoted by $\bar{\rho}$.

By Figure A-1, it can be seen that $|\hat{v} \times \bar{\rho}| = a$, where the notation $|\hat{v} \times \bar{\rho}|$ denotes the magnitude of the vector $\hat{v} \times \bar{\rho}$. Also, $\hat{v} \times \bar{\rho}$ has the direction of $\bar{q}_1$. Consequently,

$$\bar{q}_1 = \hat{v} \times \bar{\rho} \sin\chi$$

The magnitude of $\bar{q}_2$ is

$$q_2 = a(1 - \cos\chi)$$

The direction of $\bar{q}_2$ is that of the vector $\hat{v} \times (\hat{v} \times \bar{\rho})$. Also, $|\hat{v} \times (\hat{v} \times \bar{\rho})| = a$. Therefore,

$$\bar{q}_2 = \hat{v} \times (\hat{v} \times \bar{\rho})(1 - \cos\chi)$$

Since the displacement vector of particle P is $\bar{q} = \bar{q}_1 + \bar{q}_2$,

$$\bar{q} = \hat{v} \times \bar{\rho} \sin\chi + \hat{v} \times (\hat{v} \times \bar{\rho})(1 - \cos\chi) \tag{A-1}$$

The vector triple product in Eq. (A-1) may be expanded by means of the identity

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B} \, \bar{A} \cdot \bar{C} - \bar{C} \, \bar{A} \cdot \bar{B}$$

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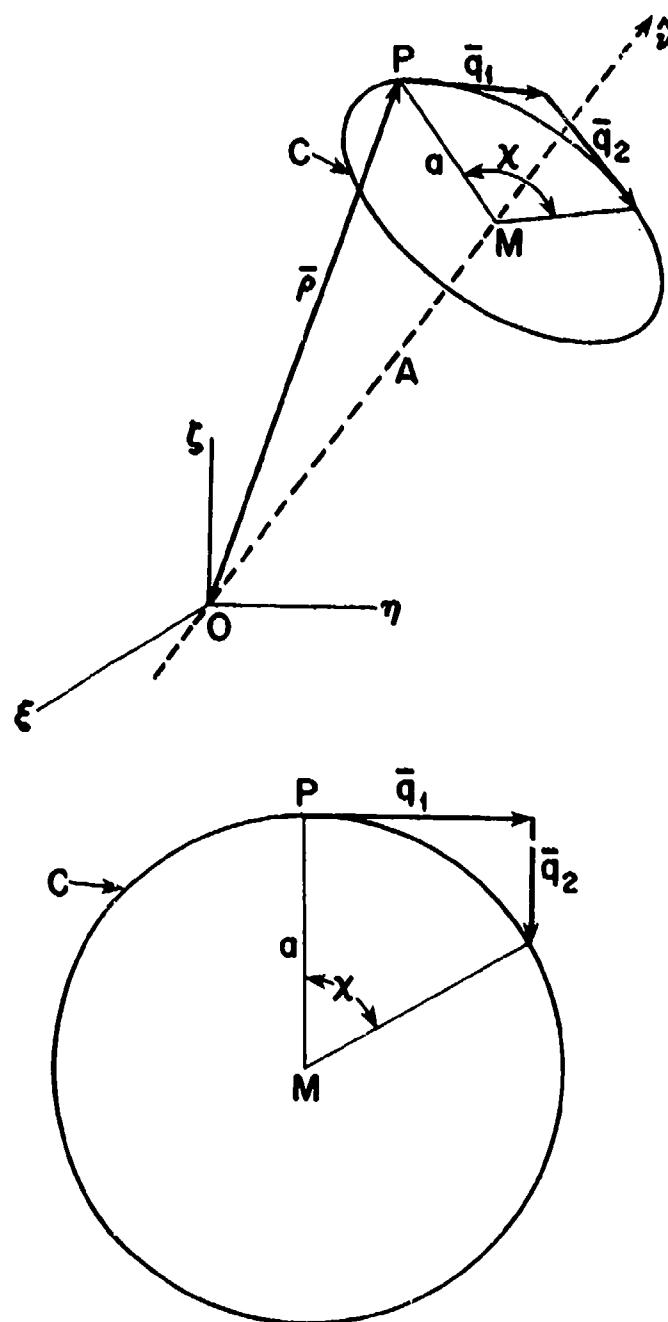
Figure A-1. Angular Displacement of Rigid Body

The particle P is displaced from position $\bar{\rho}$ to the position $\bar{R} = \bar{\rho} + \bar{q}$.
Consequently,

$$\bar{R} = \hat{\nu} \times \bar{\rho}\sin\chi + \bar{\rho}\cos\chi + \hat{\nu}\hat{\nu} \cdot \bar{\rho}(1 - \cos\chi) \tag{A-2}$$

APPENDIX B
MOMENTUM PRINCIPLES

An arbitrary mechanical system is referred to a Galilean reference frame. The momentum of the system is

$$\overline{G} = \int \overline{v} dm$$

where $\overline{v}$ is the velocity of the mass particle dm, and the integral extends throughout the system (Figure B-1). Also, $\overline{v} = d\overline{r}/dt$, where $\overline{r}$ is the radius vector from the origin 0 to particle dm (Figure B-1). Consequently,

$$\overline{G} = \frac{d}{dt} \int \overline{r} dm$$

Furthermore,

$$\int \overline{r} dm = m\overline{r}_0$$

where m is the mass of the entire system, and $\overline{r}_0$ is the radius vector from point 0 to the center of mass of the system. Therefore,

$$\overline{G} = m \frac{d\overline{r}_0}{dt} = m\overline{v}_0 \qquad\qquad (B-1)$$

where $\overline{v}_0$ is the velocity of the center of mass of the system.

The angular momentum of the system about point 0 is[2]

$$\overline{H}_0 = \int \overline{r} \times \overline{v} dm$$

Likewise, the angular momentum of the system about another point P is

$$\overline{H}_p = \int \overline{\rho} \times \overline{v} dm$$

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[2]H. L. Langhaar and A. P. Boresi, *Engineering Mechanics-Dynamics (U)*, McGraw-Hill, New York, 1959, Art. 15-10.
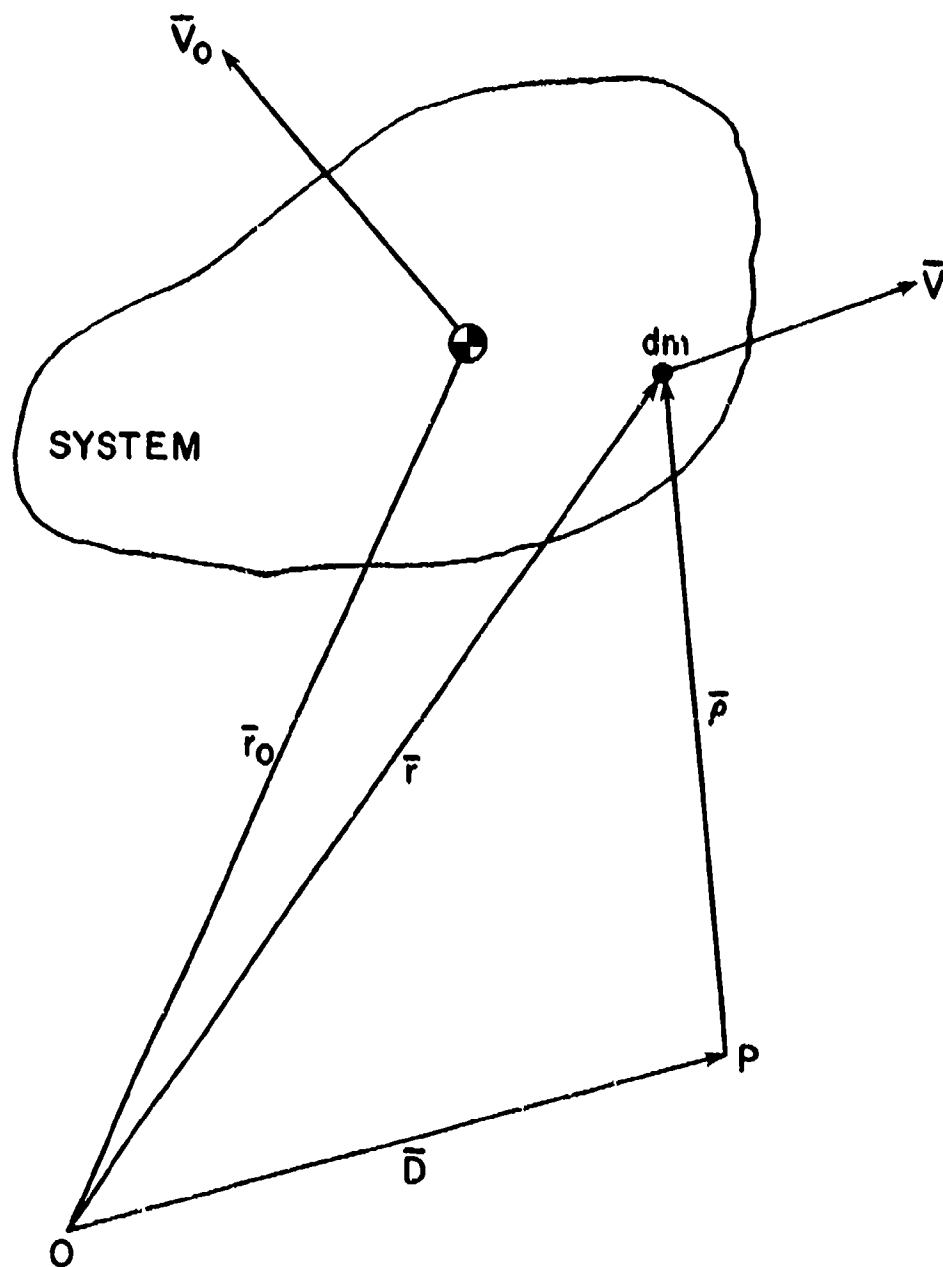
Figure B-1. Arbitrary Mechanical System

where $\bar{\rho}$ is the vector from point P to the mass particle dm. By Figure B-1, $\bar{r} = \bar{D} + \bar{\rho}$ where $\bar{D}$ is the vector $\vec{OP}$. Consequently,

$$\bar{H}_P = \int (\bar{r} - \bar{D}) \times \bar{v}\,dm = \int \bar{r} \times \bar{v}\,dm - \bar{D} \times \int \bar{v}\,dm$$

Therefore,

$$\bar{H}_P = \bar{H}_0 - \bar{D} \times \bar{G} \tag{B-2}$$

Equations (B-1) and (B-2) yield

$$\bar{H}_P = \bar{H}_0 - m\bar{D} \times \bar{v}_0 \tag{B-3}$$

Equation (B-3) serves to transfer the angular momentum from one reference point O to another reference point P.

If the system is free from external force, $\bar{H}_0$ and $\bar{G}$ are constant vectors. Then, by Eq. (B-2), $\bar{H}_P$ is a constant vector. In particular, if $\bar{H}_0 = 0$ and $\bar{G} = 0$, it follows that $\bar{H}_P = 0$, where P is any point.

APPENDIX C
REMARKS ON VECTOR ANALYSIS

In this Appendix, a brief treatment of vector analysis is presented. For more details, see Reference 2, Chapters 1, 6, 8 and 15. In this report, a letter with a bar over it denotes a vector. For example, $\bar{F}$ stands for the direction and magnitude of a force, although it does not signify the point at which the force acts. Ordinarily, the point of action of a force is designated by a statement. By definition, the letter F represents* only the magnitude of vector $\bar{F}$. Consequently, F is a non-negative number. A letter with a caret over it denotes a vector of unit magnitude. For example, $\hat{F}$ designates the direction of force $\bar{F}$.

The vector equation $\bar{A} = \bar{B}$ signifies that vectors $\bar{A}$ and $\bar{B}$ have the same magnitude and the same direction, but not necessarily the same point of action. The vector $-\bar{F}$ is defined to have the same magnitude as vector $\bar{F}$, but the opposite direction. The vectors $\bar{F}$ and $-\bar{F}$ need not have the same point of action. For example, if $\bar{F}$ denotes a force that acts on a body, the reaction of the force is $-\bar{F}$.

If k is a positive number, the product $k\bar{F}$ is defined to be a vector with the direction of $\bar{F}$, and with magnitude kF. If k is a negative number, the product $k\bar{F}$ is defined to be a vector with the direction of $-\bar{F}$, and with magnitude $|k|F$, where $|k|$ is the absolute value of k. In view of these definitions, $\bar{F} = F\hat{F}$.

The resultant $\bar{F}$ of two vectors $\bar{F}_1$ and $\bar{F}_2$ is called the sum of the vectors. The process of obtaining the sum or resultant of two vectors by the well-known parallelogram construction is called vector addition. Symbolically, $\bar{F} = \bar{F}_1 + \bar{F}_2 = \bar{F}_2 + \bar{F}_1$. It is to be observed that the relation $\bar{F} = \bar{F}_1 + \bar{F}_2$ does not imply that $F = F_1 + F_2$. In general, $F_1 + F_2$ is greater than F, since the three vectors $\bar{F}$, $\bar{F}_1$, $\bar{F}_2$ form the sides of a triangle.

By repeated applications of the parallelogram construction, one obtains the polygon construction for the sum of a number of vectors. For example, if forces $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$, $\bar{F}_4$ act at a point P of a body, their resultant $\bar{F}$ is obtained by arranging the vectors $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$, $\bar{F}_4$ in a chain,

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This convention is not always used in this report. For example, M denotes the moment of a force, but M stands for the mass of the gun. No confusion should occur, since notations are explained.

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maintaining the proper directions of the vectors. Then the line segment from the initial point P to the terminal point of the chain represents the resul nt force, Figure C-1. Symbolically, $\overline{F} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$. Vector addition is commutative and associative; i.e., the order of vectors $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$, $\overline{F}_4$ in the polygon is irrelevant, and any subset of the vectors in the polygon may be replaced by their resultant. Symbolically, this means that the vectors in the sum $\overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4$ may be permuted in any way, and parentheses may be introduced arbitrarily in the sum.

Subtraction of a vector is defined to be addition of the negative vector, i.e., $\overline{A} - \overline{B} = \overline{A} + (-\overline{B})$. Accordingly, to subtract a vector $\overline{B}$ from a vector $\overline{A}$, we reverse the direction of $\overline{B}$, and add the reversed vector to $\overline{A}$.

A direction can be defined only with respect to some reference frame. Consequently, vector analysis cannot be entirely divorced from coordinate systems. For definiteness, only right-handed systems of coordinates are considered. f (x,y,z) is a right-handed system of rectangu coordinates, the thumb, the forefinger, and the middle finger of the right hand can be directed in the positive senses along the x, y, and z axes, respectively. Frequently it is convenient to designate the directions of rectangular coordinate axes (x,y,z) by three unit vectors $(\hat{i}, \hat{j}, \hat{k})$ that coincide in direction with these axes, Figure C-2. Then the position vector $\overline{R}$ from the origin to the point P:(x,y,z) satisfies the vector equation,

$$\overline{R} = \hat{i}x + \hat{j}y + \hat{k}z \tag{C-1}$$

The orthogonal projections (F_1, F_2, F_3) of any vector $\overline{F}$ on the axes (x,y,z) satisfy the equation

$$\overline{F} = \hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3 \tag{C-2}$$

Addition and subtraction of vectors, and multiplication of vectors by scalars conform to the axioms of elementary algebra. Consequently, cert operations with vectors can be performed as in scalar algebra. For example, algebraic reduction of the vector equation

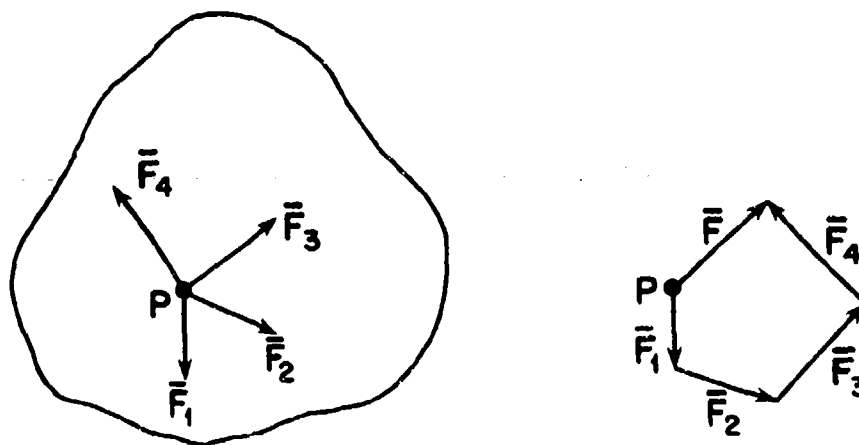$$\overline{F} = 3[\overline{A} - 2(\overline{B} - \overline{A}) + 5\overline{A} - 7\overline{B}]$$
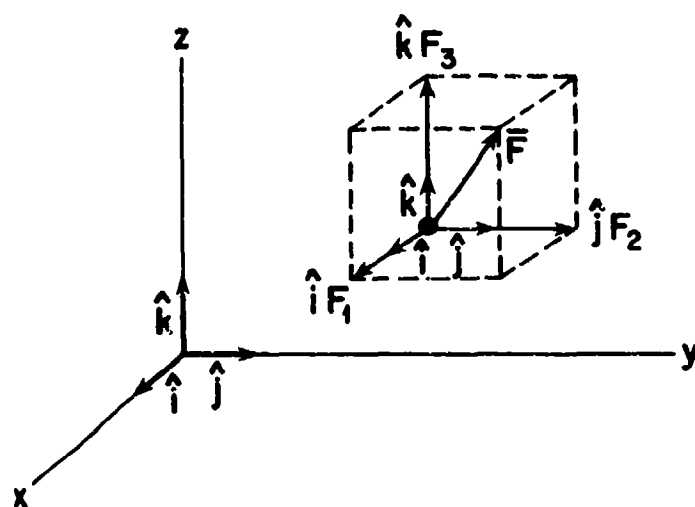
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Figure C-1. Resultant Force $\bar{F}$

Figure C-2. Resolution of Vector into Components Parallel to Coordinate Axes

yields $\overline{F} = 24\overline{A} - 27\overline{B}$.

The angle between a vector and a cartesian coordinate axis is called the direction angle of the vector with respect to the coordinate axis. The three direction angles (α, β, γ) of a vector with respect to three rectangular cartesian axes (x,y,z) determine the direction of the vector, Figure C-3. The direction angles of a vector are specified to lie in the range $0°$ to $180°$, inclusive. Consequently, a direction angle is determined uniquely by its cosine. If the cosine is negative, the angle is greater than $90°$. The cosines of the direction angles of a vector are called the direction cosines of the vector.

If (α, β, γ) are the direction angles of a vector $\overline{F}$, and if F denotes the magnitude of $\overline{F}$,

$$F_1 = F\cos\alpha \ , \ F_2 = F\cos\beta \ , \ F_3 = F\cos\gamma \tag{C-3}$$

A direction in space may be designated by a unit vector. If $\overline{F}$ is a unit vector, $F = 1$. Accordingly, Eq. (C-3) shows that the orthogonal projections of a unit vector on the x, y, and z axes are identical to the direction cosines of the vector.

By trigonometry,

$$F^2 = F_1{}^2 + F_2{}^2 + F_3{}^2 \tag{C-4}$$

Equations (C-3) and (C-4) yield

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{C-5}$$

Let $\overline{A}$ and $\overline{B}$ be two vectors whose projections on rectangular cartesian axes (x,y,z) are (A_1,A_2,A_3) and (B_1,B_2,B_3). The expression $A_1B_1 + A_2B_2 + A_3B_3$ is called the scalar product (or dot product) of the two vectors. This expression is conventionally denoted by $\overline{A} \cdot \overline{B}$. It is seen by this definition that $\overline{A} \cdot \overline{B} = \overline{B} \cdot \overline{A}$. Accordingly, scalar multiplication of vectors is said to be commutative. The importance of the scalar product arises from a geometric identity that is expressed by the equation,

$$\overline{A} \cdot \overline{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = AB\cos\theta \qquad\qquad (C-6)$$

where A and B are the magnitudes of vectors $\overline{A}$ and $\overline{B}$, and θ is the angle between these vectors.

Several special cases are to be noted. If $\overline{A} = \overline{B}$, Eq. (C-6) yields $A^2 = A_1{}^2 + A_2{}^2 + A_3{}^2$, which is equivalent to Eq. (C-4). If $\overline{B}$ is a unit vector (B = 1), it is apparent from Eq. (C-6) that $\overline{A} \cdot \overline{B}$ is the orthogonal projection of vector $\overline{A}$ on a line with the direction and sense of vector $\overline{B}$. If $\overline{A}$ and $\overline{B}$ are both unit vectors, their projections on the (x,y,z) axes are identical to their direction cosines. Hence,

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = \cos\theta \qquad\qquad (C-7)$$

where $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are the direction angles of vectors $\overline{A}$ and $\overline{B}$, and θ is the angle between these vectors. If $\overline{A} \neq 0$ and $\overline{B} \neq 0$, but $\overline{A} \cdot \overline{B} = 0$, vectors $\overline{A}$ and $\overline{B}$ are perpendicular to each other ($\theta = 90^\circ$).

Occasionally an expression of type $\overline{A}(\overline{B} \cdot \overline{C})$ arises. The parentheses may be removed; i.e.,

$$\overline{A}(\overline{B} \cdot \overline{C}) = \overline{A}\ \overline{B} \cdot \overline{C}$$

There is no ambiguity in the expression $\overline{A}\ \overline{B} \cdot \overline{C}$, since no meaning is here assigned to the expression $\overline{A}\ \overline{B}$ standing alone. Hence, $\overline{A}\ \overline{B} \cdot \overline{C}$ is a vector with the direction of vector $\overline{A}$ and with magnitude $ABC\cos\theta$, where θ is the angle between vectors $\overline{B}$ and $\overline{C}$.

Another expression that sometimes arises is

$$\hat{i}\hat{i} \cdot \overline{A} + \hat{j}\hat{j} \cdot \overline{A} + \hat{k}\hat{k} \cdot \overline{A}$$

Since the (x,y,z) components of $\overline{A}$ are $A_1 = \hat{i} \cdot \overline{A}$, $A_2 = \hat{j} \cdot \overline{A}$, and $A_3 = \hat{k} \cdot \overline{A}$, this reduces to

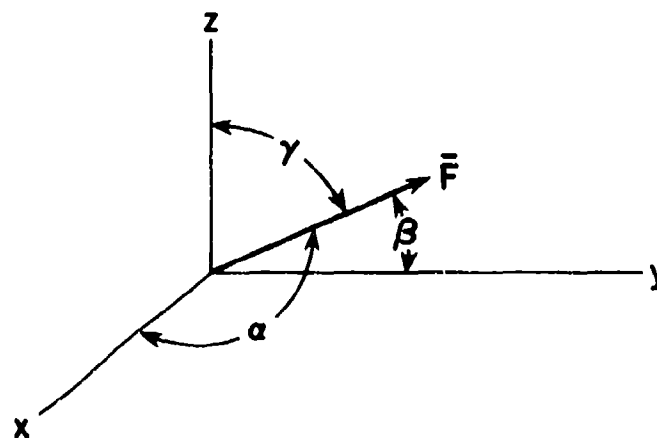$$\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3 = \overline{A}$$
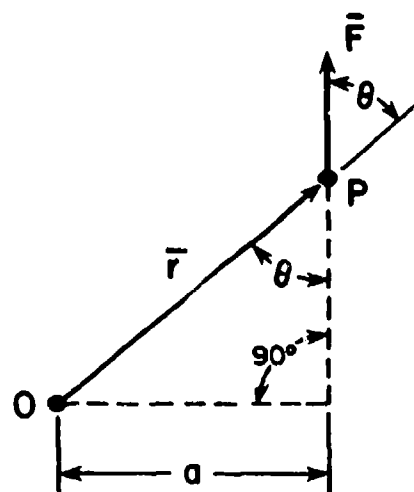
Figure C-3. Direction Angles of a Vector

Figure C-4. Moment of Force, $\bar{M} = \bar{r} \times \bar{F}$

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Consequently,

$$\hat{i}\hat{i} \cdot \overline{A} + \hat{j}\hat{j} \cdot \overline{A} + \hat{k}\hat{k} \cdot \overline{A} = \overline{A} \qquad (C-8)$$

This relation is an identity.

By cartesian expansion, it is easily seen that

$$\overline{A} \cdot (\overline{B} + \overline{C}) = \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{C}$$

Since also $\overline{A} \cdot \overline{B} = \overline{B} \cdot \overline{A}$, the scalar product conforms to the rules of elementary algebra.

From two given vectors, $\overline{A}$ and $\overline{B}$, a third vector $\overline{C}$ may be derived by the definition,

$$C_1 = A_2 B_3 - A_3 B_2 , \quad C_2 = A_3 B_1 - A_1 B_3 , \quad C_3 = A_1 B_2 - A_2 B_1 \qquad (C-9)$$

This may be expressed concisely in determinant notation:

$$\overline{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \overline{A} \times \overline{B} \qquad (C-10)$$

The vector $\overline{C}$ is called the _vector product_ or _cross product_ of $\overline{A}$ and $\overline{B}$. It may be shown by geometry that the magnitude of $\overline{C}$ is

$$C = AB\sin\theta \qquad (C-11)$$

where θ is the angle between vectors $\overline{A}$ and $\overline{B}$. It follows from Eq. (C-9) that $\overline{C} \cdot \overline{A} = \overline{C} \cdot \overline{B} = 0$. Consequently, vector $\overline{C}$ is perpendicular to both of the vectors $\overline{A}$ and $\overline{B}$. It can be shown that, if the coordinates (x,y,z) are right-handed, the sense of vector $\overline{C}$ is that in which a right-hand screw advances when turned from $\overline{A}$ to $\overline{B}$.

The vector product is not commutative. Since a permutation of two rows in a determinant changes the sign of the determinant, Eq. (C-10)

shows that $\overline{B} \times \overline{A} = -\overline{A} \times \overline{B}$. In spite of this anomalous behavior, the vector product has other properties in common with ordinary multiplication. In particular,

$$\overline{R} \times (\overline{A} + \overline{B}) = \overline{R} \times \overline{A} + \overline{R} \times \overline{B} \; , \quad (\overline{A} + \overline{B}) \times \overline{R} = \overline{A} \times \overline{R} + \overline{B} \times \overline{R}$$

$$(C-12)$$

Hence,

$$(\overline{A} + \overline{B}) \times (\overline{C} + \overline{D}) = \overline{A} \times \overline{C} + \overline{A} \times \overline{D} + \overline{B} \times \overline{C} + \overline{B} \times \overline{D}$$

It is seen from Eq. (C-11) that the vector product of two parallel vectors is zero, since $\theta = 0$. Hence,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

Also, by Eq. (C-10)

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{k}$$

Similarly, $\hat{k} \times \hat{i} = \hat{j}$ and $\hat{j} \times \hat{k} = \hat{i}$. Evidently,

$$\overline{A} \times \overline{B} = (\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3) \times (\hat{i}B_1 + \hat{j}B_2 + \hat{k}B_3)$$

Algebraic expansion of the right side of this equation leads back to Eq. (C-9).

The expression $\overline{A} \cdot (\overline{B} \times \overline{C})$ is called the <u>scalar triple product</u>. It may be written without parentheses as $\overline{A} \cdot \overline{B} \times \overline{C}$, since $(\overline{A} \cdot \overline{B}) \times \overline{C}$ has no meaning. The expression $\overline{A} \cdot \overline{B} \times \overline{C}$ is a scalar. Cartesian expansion yields the determinant form,

$$\overline{A} \cdot \overline{B} \times \overline{C} = \begin{array}{ccc} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{array} \qquad (C-13)$$

Since a transposition of two rows in a determinant merely changes the sign of the determinant,

$$\overline{A} \cdot \overline{B} \times \overline{C} = \overline{B} \cdot \overline{C} \times \overline{A} = \overline{C} \cdot \overline{A} \times \overline{B} \qquad (C-14)$$

The absolute value of $\overline{A} \cdot \overline{B} \times \overline{C}$ represents the volume of the parallelepiped having concurrent edges represented by $\overline{A}$, $\overline{B}$, and $\overline{C}$.

The vector triple product is $\overline{A} \times (\overline{B} \times \overline{C})$. The parentheses are essential in this expression. By cartesian expansion, the following identity can be verified:

$$\overline{A} \times (\overline{B} \times \overline{C}) = \overline{B}\, \overline{A} \cdot \overline{C} - \overline{C}\, \overline{A} \cdot \overline{B} \qquad (C-15)$$

This may be memorized as the "Back-Cab" formula.

The vector product is useful for representing moments of forces. Let a be the perpendicular distance from a given point O to the line of action of a given force $\overline{F}$, Figure C-4. The moment of force $\overline{F}$ about point O is defined to be a vector $\overline{M}$ with magnitude Fa. The vector $\overline{M}$ is defined to be perpendicular to the plane determined by the force $\overline{F}$ and point O. The sense of vector $\overline{M}$ is defined by the right-hand-screw rule; i,e., vector $\overline{M}$ points in the direction that a right-hand screw would advance if force $\overline{F}$ should cause it to turn about an axis through point O. For the case illustrated by Figure C-4, vector $\overline{M}$ is directed toward the reader, perpendicular to the plane of the paper. Let the vector OP (Figure C-4) be denoted by $\overline{r}$. Then the conditions of the preceding definition are fulfilled by the vector equation,

$$\overline{M} = \overline{r} \times \overline{F} \qquad (C-16)$$

The moment of a force about an axis is a scalar. If $\hat{n}$ is a unit vector in the direction of axis L, the moment of force $\overline{F}$ about axis L is $M_L = \hat{n} \cdot \overline{M}$, where $\overline{M}$ is the moment of force $\overline{F}$ about any point O on axis L. Consequently, by Eq. (C-16),

$$M_L = \overline{F} \cdot \hat{n} \times \overline{r} = \overline{r} \cdot \overline{F} \times \hat{n} = \hat{n} \cdot \overline{r} \times \overline{F} \qquad (C-17)$$

An infinitesimal increment $d\overline{R}$ of a vector $\overline{R}$ need not be collinear with the vector $\overline{R}$, Figure C-5. Consequently, in general, the vector $\overline{R} + d\overline{R}$ differs from the vector $\overline{R}$, not only in magnitude, but also in direction. It would be misleading to denote the magnitude of vector $d\overline{R}$ by dR, since dR denotes the increment of the scalar R. Accordingly, the magnitude of $d\overline{R}$ is denoted by $|d\overline{R}|$, or by another symbol, such as ds. The magnitude of the vector $\overline{R} + d\overline{R}$ is R + dR. Figure C-5 shows that $dR \leq |d\overline{R}|$. If the vector $\overline{R}$ is a function of a scalar t (where t may or may not denote time), $d\overline{R}/dt$ is defined to be a vector in the direction of $d\overline{R}$ with magnitude ds/dt, where $ds = |d\overline{R}|$. If $\overline{R}$ is the position vector of a particle, and if t denotes time, $\overline{v} = d\overline{R}/dt$ is the velocity of the particle and $d\overline{v}/dt = d^2\overline{R}/dt^2$ is the acceleration of the particle. Vectors obey the same rules of differentiation as scalars. This may be shown by the delta method that is used for deriving differentiation formulas in scalar calculus. For example, if $\overline{Q} = u\overline{R}$, where u is a scalar function of t and $\overline{R}$ is a vector function of t,

$$\frac{d\overline{Q}}{dt} = \dot{\overline{Q}} = \dot{u}\overline{R} + u\dot{\overline{R}}$$

in which the dot denotes the derivative with respect to t. Likewise,

$$\frac{d}{dt}(\overline{A} \cdot \overline{B}) = \dot{\overline{A}} \cdot \overline{B} + \overline{A} \cdot \dot{\overline{B}}$$

and $$\frac{d}{dt}(\overline{A} \times \overline{B}) = \dot{\overline{A}} \times \overline{B} + \overline{A} \times \dot{\overline{B}}$$

The angular velocity $\overline{\omega}$ of a rigid body is a vector quantity. For, let $\overline{\omega}$ represent the angular velocity of a rigid body whose motion at the

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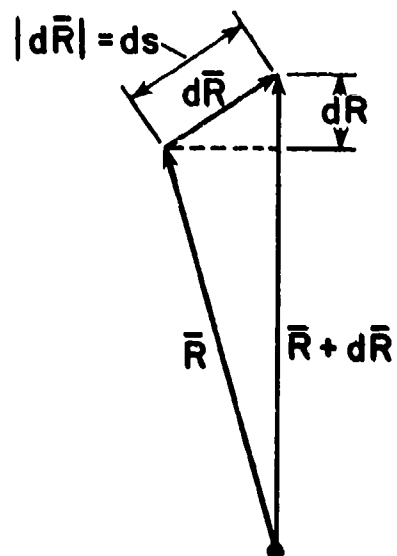
Figure C-5. Infinitesimal Increment $d\bar{R}$
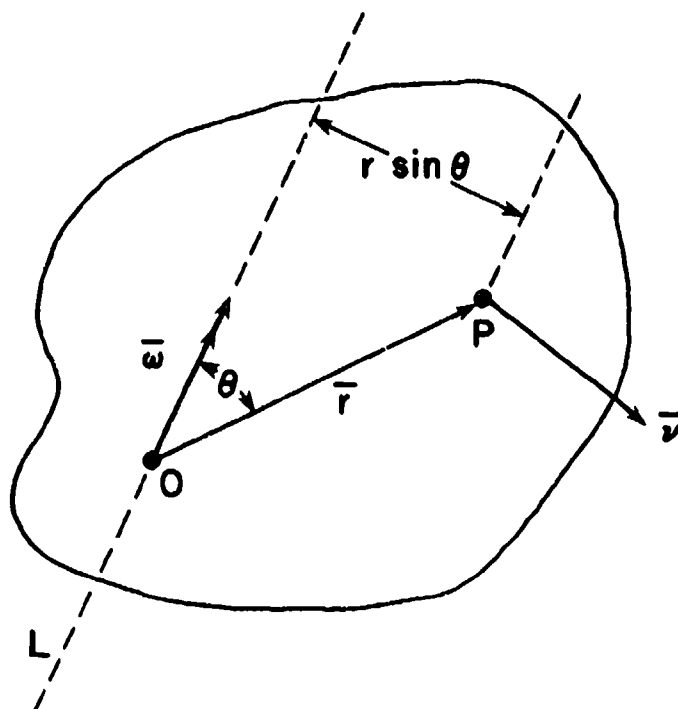
Figure C-6. Velocity $\bar{v} = \bar{\omega} \times \bar{r}$

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instant under consideration is a rotation about an axis L. Let $\bar{r}$ be the vector from any point O on axis L to a particle P of the body (Figure C-6). The distance of particle P from the axis of rotation is $r\sin\theta$, where θ is the angle between the vectors $\bar{r}$ and $\bar{\omega}$. Hence, the speed of particle P is $r\omega\sin\theta$. The velocity $\bar{v}$ of particle P is perpendicular to the plane of vectors $\bar{r}$ and $\bar{\omega}$, and its sense is determined by the right-hand-screw rule (if right-handed coordinates are used). Therefore,

$$\bar{v} = \bar{\omega} \times \bar{r} \tag{C-18}$$

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